KAKATIYA GOVERNMENT COLLEGE

HANUMAKONDA

Name : Dr. D. Venkatesh
Designation : Assistant Professor
Year of Award of PhD. : 2023
Name of the University : Osmania University
Year of entering into Govt. Service : 2011

S. No.	Details of copies of Certificates	
1	Copy of Ph.D Certificate	enclosed
2	Press note	enclosed
3	Research work dates of seminars and Pre-Ph.D Date of joining in this college	enclosed
4	Details of Ph.D Admission-part time or full time	enclosed -(Part time)
5	Copies of RDC Approval letters of Ph.D	— NA -
6	Name of guide/supervisors with mobile number, email id	enclosed
7	Copies of guide allotment letter	enclosed
8	No. of increments sanctioned for Ph.D.	03
9	Published Research article-copies.	enclosed-
10	Original Ph.D Thesis.- Book.	Available at office ~~enclosed~~

Signature

Name & Designation

Dr. D. Venkatesh
Assistant Professor
Dept. of Mathematics.

PRINCIPAL
KAKATIYA GOVT. COLLEGE
Hanamkonda.

OU 969515

Osmania University

Faculty of Science

This is to certify that D Venkatesh

son / daughter of D Narsaiah

having pursued a course of study prescribed by this University and having passed the requirements by Examination and by thesis has been admitted to the Degree of

Doctor Of Philosophy

in the Subject of Mathematics

The title of the Thesis is :

Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces

The candidate has been declared qualified for the award of the Degree of Ph.D.on 06 Oct 2023

Given under the seal of the University

CN102380811

Hyderabad, T.S.

Dated Kartika 9,1945
October 31,2023

Vice-Chancellor

CONFIDENTIAL SECTION
EXAMINATION BRANCH
NO. 781/Ph.D/Exams/2023

OSMANIA UNIVERSITY
HYDERABAD-500 007,T.S.
Dated: 06 Oct, 2023

PRESS NOTE

The following candidates who had presented the Thesis on the subject mentioned against each for the degree of Ph.D are declared qualified for the award of Degree of Doctor of Philosophy (Ph.D.) of Osmania University, Hyderabad.

Ph.D.

S.N	Reference No.	Name of the Candidate/ Father Name	Subject	Thesis Title	Supervisor/ Regn. Date
1	PHD44434	**Mr. Gardas Naveen Kumar** S/o. Gardas Krishna	Applied Geo-Chemistry	Hydrogeochemical, Geophysical Investigations for Groundwater Quality Impact of Limestone Mining Areas, Using Remote Sensing and GIS in and Around Tandur, Vikarabad District, Telangana State, India	Prof. B Srinivas Dept. of App. Geochemistry, O.U., Hyd. **(23/03/2017)**
2	PHD44435	**Mr. Touseefahamad Mohammad** S/o. Mohammadrazack	Mechanical Engineering	Modeling of Thermal Runaway of Dinitrotoluene to Trinitrotoluene, Ammonium Nitrate Decomposition and Trinitrotoluene Equivalence	Dr. A Seshu Kumar Chief Scientist, CSIR-IICT, Hyd./Co-Supervisor: Prof. A M K Prasad(Retd.), Dept. of Mech. Engg., O.U., Hyd. **(04/03/2013)**
3	PHD44436	**Mr. D Venkatesh** S/o. D Narasaiah	Mathematics	Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces	Prof. V Nagaraju Dept. of Mathematics, O.U., Hyd. **(28/03/2017)**
4	PHD44437	**Ms. B Vijayalakshmi** D/o. B Venkataiah	Chemical Engineering	Studies on Preparation and Characterization of Wood-Plastic Composites Using Recycled Expanded Polystyrene and Biomass Materials: Sustainable Management Approach	Prof. E Nagabhushanam(Retd.) Univ. Coll. of Tech., O.U., Hyd. **(23/03/2017)**
5	PHD44438	**Mr. Vuba Kiran Kumar** S/o. V Venkanna	Chemical Engineering	Combined Effect of MWCNT & Various Types of Maleic Anhydride Grafted Polypropylene on Mechanical & Thermal Properties of Polypropylene Co-Polymer	Prof. E Nagabhushanam(Retd.) Univ. Coll. of Tech., O.U., Hyd. **(30/03/2017)**
6	PHD44439	**Mr. Chandiri Venkateshwar Reddy** S/o. Chandiri Gal Reddy	Mechanical Engineering	Study and Characterization of Epoxy Based Composite Materials for High Strength Applications	Prof. M Ramesh Babu Dept. of Mech. Engg. O.U., Hyd./Co-Supervisor: Dr. R Ramanarayanan(Retd.), DRDO, Hyd. **(13/02/2013)**

Ch. Vardhani 6/10/23

Addl.Controller of Examinations
(Confidential)

Dr. N. Kishan
Professor & Head

Department of Mathematics
Osmania University,
HYDERABAD- 500007.
Phone : 27095083, 27682389.
Email: headmathsou@gmail.com

Date: 10-04-2023

CERTIFICATE

This is to certify that **Mr. Duduka Venkatesh,** Research Scholar working under the supervision of **Prof.V.Nagaraju** has given the Ph.D. Third Pre-submission Seminar talk on his Ph.D. work **"EXISTENCE AND UNIQUENESS OF FIXED AND COMMON FIXED POINTS FOR DIFFERENT CONTRACTION IN VARIOUS SPACES"** on 10-04-2023 at 2:00 p.m. in the Department of Mathematics.

CHAIRPERSON
BOS in Mathematics
Dept. of Mathematics, OU.

Chairperson
BoS in Mathematics
Department of Mathematics
Osmania University
Hyderabad-500 007.

HEAD

Head
Department of Mathematics
Osmania University
Hyderabad-500 007.

Dr. N. Kishan
Professor & Head

Department of Mathematics
Osmania University,
HYDERABAD- 500007.
Phone : 27095083, 27682389.
Email: headmathsou@gmail.com

Date: 10-03-2023

CERTIFICATE

This is to certify that **Mr. Duduka Venkatesh,** Research Scholar working under the supervision of **Prof. V. Nagaraju** has given the Ph.D. Second Research Progress Seminar talk on his Ph.D. work **"FIXED POINT THEOREMS"** on 10-03-2023 at 11:00 a.m. in the Department of Mathematics.

CHAIRPERSON
BOS in Mathematics
Dept. of Mathematics, OU.
Chairperson
BoS in Mathematics
Department of Mathematics
Osmania University
Hyderabad-500 007.

HEAD
Head
Department of Mathematics
Osmania University
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Dr. N. Kishan
Professor & Head

Department of Mathematics
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Phone : 27095083, 27682380.
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Date: 19-02-2022

CERTIFICATE

This is to certify that **Mr. D. Venkatesh**, Research Scholar working under the supervision of **Dr. V. Naga Raju** has given the **Ph.D. Research Design Seminar (First)** on his Ph.D. work entitled **"FIXED POINT THEOREMS"** on **19-02-2022** at **12-00 noon.** in the Department of Mathematics.

19.02.2022
CHAIRPERSON
BOS in Mathematics

Chairperson
BoS in Mathematics
Department of Mathematics
Osmania University,
Hyderabad-500 007.

19.02.202
HEAD

Head
Department of Mathematics
Osmania University
Hyderabad-500 007.

6/17
PA

OSMANIA UNIVERSITY

MEMORANDUM OF MARKS

PA 256109

Examination Ph.D. COURSE WORK SEPTEMBER 2018

REF NO. : 201927405 FACULTY OF SCIENCE DATE : 26-06-2019

NAME : DUDUKA VENKATESH ROLL NO. : 900717541021

PARENT(S) NAME : D NARSAIAH

SL. NO.	SUBJECT NAME MATHEMATICS	UNIVERSITY EXAMINATION MAXIMUM MARKS	UNIVERSITY EXAMINATION MARKS SECURED	RESULT
1	RESEARCH METHODOLOGY	100	50	PASS
2	SPECIALISATION (BROAD FIELD)	100	73	PASS
	TOTAL	200	123	
	GRAND TOTAL	===	===	

TOTAL IN WORDS : * ONE * TWO * THREE *

GRAND TOTAL AT THE END OF THE COURSE : =====

RESULT : PASSED

MINIMUM PASS MARKS:FIFTY

CLERK-INCHARGE SUPERINTENDENT CONTROLLER OF EXAMINATIONS

(12) ②

JOINING REPORT OF Ph.D. COURSE, FACULTY OF SCIENCE, OSMANIA UNIVERSITY

1	Name & Phone No / Male / Female	:	D.VENKATESH & 9985185643/Male
2	Father's Name	:	D.NARSAIAH
3	Details of Scholarship if any	:	-
4	College/Institute at which the Candidate proposes to work	:	UCE, OSMANIA UNIVERSITY
5	Full-Time/Part-Time	:	PART TIME
6	Name of the Supervisor	:	Sri Dr. V. Nagaraju
7	Department	:	Mathematics
8	State wheither you being to OC/BC (A/B/C/D/E)SC/ST	:	BC-B
9	Topic of Research	:	Fixed Point Theorems

To
The Dean,
Faculty of Science

//Through Proper Channel//

Sir,

Ref: Order No: 3498/DFSc/OU/2017 Dt 10-03-2017

I am herewith submitting my joining report today i.e. on 28-03-2017

I have read the rules and regulations of the Ph.D. Course/ Course and I undertake to abide by them.

I understand that my admission may be cancelled, if the statements I made in my application are found to be false.

I have satisfied all conditions stipulated in my admission order and I am enclosing herewith the necessary certificates (if applicable).

LIST OF ENCLOSURES

1. D.D. No: 121157 Date 28-03-2017 Amount 2000=00

Dr. V. NAGARAJU
ASST. PROFESSOR
DEPT. OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING
OSMANIA UNIVERSITY, HYDERABAD - 500 007.

SIGNATURE OF SUPERVISOR

SIGNATURE OF CANDIDATE

SIGNATURE OF THE HEAD OF INSTITUTION IN WHICH CANDIDATE PROPOSES TO WORK

PRINCIPAL,
Univ. College of Engg; O.U.

30/3/17

SIGNATURE OF THE HEAD OF THE UNIV.DEPARTMENT

Head
Department of Mathematics
Osmania University
Hyderabad - 500 007

SIGNATUTRE OF THE DEAN

DEAN
Faculty of Science
OSMANIA UNIVERSITY,
HYDERABAD-500 007.

ANNEXURE – III

PROFORMA TO BE FILLED BY THE CANDIDATE SELECTED UNDER THE CATEGORY **"SUBJECT TO TAKING LEAVE AS PER RULES"**

I, D. VENKATESH have been provisionally selected for the Ph.D. course as Part Time Research Scholar in the Faculty of Science for the academic year **2013- 2014** in the subject of Mathematics and I hereby agree that I would take leave for a minimum of one year for attending the classes of the Ph.D. work during the tenure of the Ph.D. course and a letter from the employer that the required leave will be sanctioned for the purpose stated.

HYDERABAD

SIGNATURE OF THE CANDIDATE

Date: 28-03-2017

Name: D. VENKATESH

Signature of the supervisor

Name: Dr. V. NAGARAJU
ASST. PROFESSOR
DEPT. OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING
OSMANIA UNIVERSITY, HYDERABAD - 500 007.

Seal:

Signature of the Head 30/7/17

Name Head

Seal: Department of Mathematics
Osmania University
Hyderabad - 500 007

OFFICE OF THE DEAN FACULTY OF SCIENCE

OSMANIA UNIVERSITY HYDERABAD

No. 3498 /DFSc/2017

Date: 10.03.2017

ORDERS

Sub : FACULTY OF SCIENCE, OU – Admission to Ph.D. Course Category II /2013-2014 – Orders – Biochemistry/Geophysics/Mathematics/Microbiology -Issued.

Ref: No 309 /F/Acad-III/2017 Dated 21.2.2017

* * *

The candidates in the enclosed list are provisionally admitted to the Ph.D. course of Osmania University for the academic year 2013-2014 on the recommendation of the Admission Committee in the Faculty of Science in the subject mentioned against his/her name.

The selected candidates are required to fulfill the conditions, if mentioned against their names, and to submit their Joining Reports (Proforma provided), by **10. 04 . 2017** failing which their admission orders would be deemed to have been withdrawn. No further notice will be given. The Joining Reports along with the original D.D. and all necessary documents should be submitted in the concerned Departments. **No joining report will be accepted without the T.C. (Transfer Certificate) in original or a letter from the respective University where from the Post Graduate Degree has been obtained to the effect that no separate Transfer Certificate will be issued by that University.** The Dean's office shall then issue a list of names of the admitted candidates to the Heads of the Departments concerned, which shall be final.

The registration is valid for a period of four years for Full Time Research Scholars and five years for Part Time Research Scholars from the date of joining after which period it will be cancelled unless otherwise extended.

All the selected candidates both Full-Time and Part-Time have to pay the fee as under:

1. Both Full Time and Part Time Scholars working in the Osmania University .. Rs.2000 per year
2. Scholars working in recognised Research Centres outside the University .. Rs.5000 per year

(P.T.O.)

through a demand draft in favour of "Dean, Faculty of Science, Osmania University." They should submit their Joining Reports in the concerned University Department in the prescribed proforma in triplicate along with the Original DD., and M.Sc., Certificate (Xerox Copy) in proof of satisfaction of the conditions stipulated. If the candidate fails to pay the fees mentioned above within the specified time his/her admission will be cancelled without further notice to the candidate.

The selected candidates are required to submit an undertaking to the effect that they do not ask for hostel facilities (Annexure II) along with their joining reports, failing which they will not be granted admission.

Candidates selected under the category "Part Time" are required to submit an undertaking in triplicate on the proforma provided (Annexure-III) that they would be taking necessary leave as per rules of the University.Their admission is conditional upon realization of dues to the University if any from the candidates. The admissions are made on the basis of the present occupation of the candidates. In case there is a change in occupation or place of work during the period of their candidature in the Ph.D., course, their admission is liable to be cancelled. Any change in their occupation should be brought to the notice of the Dean, through the Supervisor and the Head of the Department, and the Dean may permit the candidate to continue his/her Ph.D. course as per the rules.

The candidates who are admitted to the Ph.D. course shall not pursue any other course or appear for any other examination leading to any other Degree (both Full-Time and Part-Time) of this University or any other University. Any violation of this regulation will lead to the cancellation of admission.

DEAN

Faculty of Science, O.U.

DEAN

Faculty of Science

OSMANIA UNIVERSITY,

HYDERABAD-500 007

To

The Research Scholar concerned.

Copy forwarded for information and necessary action to:-

1. Principal, University College of Science, O.U.
2. The Vice Principal, Hostels, Univ.College of Science, O.U.
3. The Head, Department of _________________________, O.U.
4. The Controller of Examinations, O.U.
5. The Asst. Registrar (Academic),O.U.
6. The Librarian, University Library, O.U.
7. The Secretary to Vice-Chancellor, O.U.
8. The Officer on Special Duty to Vice-Chancellor, O.U.
9. The P.A. to Registrar, O.U.
10. The Chief Warden, Hostels & Messes, O.U.

LIST OF CANDIDATES SELECTED FOR ADMISSION INTO PH. D 2013-2014, CATEGORY II.

No. 3498/DFSC/OU/2017 **Date : 10/03/2017**

Ref.: 309/F/Acad-III/2017, Dt: 21-2-2017

SUBJECT: Mathematics

S. No	Name of the Candidate	Gender	Category	FT / PT	Place of Work	Name of the Supervisor
1	Mr.Bheemudu C	M	BC-A	PT	OU	Dr. S. N. Hasan
2	Mr.M.Prabhakar	M	BC-D	PT	OU	Dr. A. Venkata Lakshmi
3	Mr. L. Nagraj	M	BC-A	PT	OU	Dr. N. Kishan
4	Mr.A.Krishna Rao	M	BC-B	PT	OU	Dr. Y. Rameshwar
5	Mr.S.Raju	M	SC	PT	OU	Dr. V. Dhanalakshmi
6	Mr.Bachha Sri Sailam	M	BC-B	PT	OU	Prof. J. Anand Rao
7	Mr. V. Samba Siva Rao	M	OC	PT	OU	Prof. M. Rangamma
8	Mr.M.Venkateshwar Rao	M	OC	PT	OU	Dr. B.G. Siddharth
9	Mr.Ch.Krishna	M	SC	FT	OU	Dr. V. Srinivas
10	Mr. Bukya Ravi	M	ST	PT	OU	Dr. A. Venkata Lakshmi
11	MR. K. Santhosh Reddy	M	OC	PT	OU	Prof. M. Rangamma
12	Mr.D.Venkatesh	M	BC-B	PT	OU	Dr. B.G. Siddharth
13	Ms. T. Rajeshwari	F	BC-B	PT	OU	Prof. B. Shanker
14	Mr. Suresh Devarapalli	M	BC-D	PT	OU	Dr. S. N. Hasan
15	Mr.D.Pavan Kumar	M	BC-B	PT	OU	Prof. V. V.Hara Gopal
16	Ms.D.Srilatha	F	BC-B	PT	OU	Dr. B. Krishna Reddy
17	Mr.S.Jagadesh	M	OC	FT	OU	Dr. M. Chenna Krishna Reddy

Details of the Supervisor

Name of the student : Dr. D.Venkatesh

Name of the Guide : Prof. V. Naga Raju

Contact Number of the Guide : 9440496134

Mail ID of the supervisor: viswanag2007@gmail.com

Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 18, Number 2 (2022), pp. 753-770

http://www.ripublication.com/gjpam.htm

Some Fixed Point Outcomes in S_b-Metric Spaces using (ϕ, ψ)-Generalized Weakly Contractive Maps in S_b-Metric Spaces

Duduka Venkatesh[*1] **and V. Naga Raju**[2]

[1,2]*Department Of Mathematics, Osmania University, Hyderabad, Telangana-500007, India. , E-mail:* 1*venkat409151@gmail.com,* 2*viswanag2007@gmail.com*

Abstract

In this result, we define (ψ, ϕ) -generalized weakly contraction map in S_b-metric space. In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)-generalized weakly contractive maps in S-metric spaces and established the existence and uniqueness of fixed point theorem for such maps. By the motivation of B.K.Leta and G.V.R.Babu[3] results in S-metric spaces, we introduced the (ψ, ϕ) - generalized weakly contractive map in S_b-metric spaces and prove a existence and uniqueness of fixed point theorem. We also give an example to support of our result.

Keywords: Fixed point, S-metric space, S_b-metric space, (ψ, ϕ)- generalized weakly contracition map.
2010 MSC: 47H10, 54E50

1. INTRODUCTION

During 1922, Stefan Banach conceived the concept of contraction and established well known Banach contraction theorem. Banach Principle of contraction[9] on metric spaces is the paramount importance cause in the field of fixed points and non linear analysis. Literature's are brought out new outcomes that are related to prove the generalization of metric space and to acquire a refinement about the contractive

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condition. In the year 2006, B Sims and Mustafa[10], established theory on G-metric spaces, that is an extension of metric spaces and established some properties. Later, A.Aliouche, S.Sedghi and N.Shobe [7] initiated S-metric spaces, it is a generalization of G-metric spaces in the year 2012. In 2014, S.Radojevic, N.V.Dung and N.T.Hieu [11] proved by examples shows S-metric spaces are not a generalization of G-metric and controvisely. Recently, N.Mlaiki and N.Souayah[8] introduced the S_b-metric spaces as the generalization of b-metric spaces and S-metric spaces and proved some fixed point results were proved for such spaces in [8]. Very recently Ozur and Tas[5] studied some relations between S_b-metric spaces and some other metric spaces. Fixed points of contractive maps on S-metric spaces were studied in [2,3,7,11-15] and some fixed point results in S_b-metric space were also studied by different authors in [5,6,8].

In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)- generalized weakly contractive maps in S-metric spaces and established the existence and uniqueness of fixed point theorem for such maps. By the motivation of B.K.Leta and G.V.R.Babu[3] results in S-metric spaces, we introduced the (ψ, ϕ) - generalized weakly contractive map in S_b-metric spaces and prove a existence and uniqueness of fixed point theorem. Let us see some basic definitions, Examples and Lemmas for the sake of transparency.

2. PRELIMINARIES

Definition 2.1.[7] Let $X \neq \emptyset$, then a mapping S:$X^3 \to [0, \infty)$ is said to be an S-metric on X if:
(S1) $S(\xi, \vartheta, w) > 0$ for all $\xi, \vartheta, w \in X$ with $\xi \neq \vartheta \neq w$.
(S2) $S(\xi, \vartheta, w)$ = 0 if $\xi = \vartheta = w$.
(S3) $S(\xi, \vartheta, w) \leq [S(\xi, \xi, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$
for any $\xi, \vartheta, w, a \in X$. Then we call (X, S) is an S-metric space.
Example 2.1.[13] Suppose X=R, Collection of all real numbers and let $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X,S) becomes a S-metric space.
Definition 2.2.[5] Let $X \neq \emptyset$ and s$\geq$1. Then we say that a function
$d : X^2 \to [0, \infty)$ is a b-metric on X if
(i) $d(\xi, \vartheta) = 0 \iff \xi = \vartheta$.
(ii) $d(\xi, \vartheta) = d(\vartheta, \xi)$ for all $\xi, \vartheta \in X$.
(iii) $d(\xi, \vartheta) \leq s[d(\xi, w) + d(w, \vartheta)]$, for all $\xi, \vartheta, w \in X$.
The pair (X,d) is known as b-metric space with s$\geq$1.
Definition 2.3.[1] Let $X \neq \emptyset$ and s$\geq$1. Then we say a mapping $S_b : X^3 \to [0, \infty)$ is S_b-metric on X if :
(i) $S_b(\xi, \vartheta, w)$ = 0 if $\xi = \vartheta = w$.

(ii)$S_b(\xi, \vartheta, w) \leq s[S_b(\xi, \xi, a) + S_b(\vartheta, \vartheta, a) + S_b(w, w, a)]$
$\forall \xi, \vartheta, w, a \in X$. The pair (X, S_b) is known as S_b-metric space.
Each S-metric space is a S_b-metric space for s=1, but the converse statement is not true. We find an example of S_b-metric, but not an S-metric on X in [5].

Definition 2.4.[1] Consider (X, S_b) be a S_b-metric space for s>1. Then S_b-metric is known as symmetric if $S_b(\xi, \xi, \vartheta) = S_b(\vartheta, \vartheta, \xi), \forall \xi, \vartheta \in X$.

Lemma 2.1.[4] In S_b-metric space, we have
(i) $S_b(\xi, \xi, \vartheta) \leq sS_b(\vartheta, \vartheta, \xi)$ and $S_b(\vartheta, \vartheta, \xi) \leq sS_b(\xi, \xi, \vartheta)$
(ii) $S_b(\xi, \xi, w) \leq 2sS_b(\xi, \xi, \vartheta) + s^2 S_b(\vartheta, \vartheta, w)$.

Definition 2.5.[4] If (X, S_b) is an S_b-metric space and a sequence $\{\xi_n\}$ in X. Then
(i) $\{\xi_n\}$ is called a S_b-Cauchy sequence, if to every $\epsilon > 0$, $\exists$ $n_0 \in N$ so that $S_b(\xi_n, \xi_n, \xi_m) \leq \epsilon, \forall n, m > n_0$.
(ii) $\{\xi_n\} \to \xi \iff$ to each $\epsilon > 0$, $\exists$ $n_0 \in N$ such that $S_b(\xi_n, \xi_n, \xi) < \epsilon$ and $S_b(\xi, \xi, \xi_n) < \epsilon \; \forall n \geq n_0$, and we write as $\lim_{n \to \infty} \xi_n = \xi$.

Definition 2.6.[4] We say that (X, S_b) is complete if each S_b-Cauchy sequence is S_b-Convergent in X.

Tas and Ozgur[5] proved the following theorems in S_b-metric spaces.

Theorem 2.1.[5] Consider (X, S_b) be a complete S_b-metric space and s ≥ 1. If h is a self map on X satisfying

$$S_b(h\xi, h\xi, h\vartheta) \leq c\, S_b(\xi, \xi, \vartheta) \;\; \forall \xi, \vartheta \in X, \;\; where \;\; 0 < c < \tfrac{1}{s^2}.$$

Then h has a unique fixed point ξ in X.
In this article we indicate:
(i)$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi$ is non decreasing, continuous and ψ(t)=0 $\iff$ t=0.}
(ii) $\Phi = \{\phi : [0, \infty) \to [0, \infty)$: ϕ is continuous, $\phi(t) = 0 \iff t = 0\}$.

In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)- generalized weakly contractive maps in S-metric spaces and proved existence and uniqueness of fixed point theorem for such maps as follows.

Definition 2.7[3] Consider (X,S) be an S-metric space and h be a self map on X. Suppose that $\exists$ $\alpha \in (0, 1)$, $\psi \in \Psi$ and $\phi \in \Phi$ so that
$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(P_\alpha(\xi, \vartheta, w)) - \phi(P_\alpha(\xi, \vartheta, w))$ (2.1)
where $P_\alpha(\xi, \vartheta, w)$ = max$\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw),$
$\alpha S(h\xi, h\xi, \vartheta) + (1 - \alpha)S(h\vartheta, h\vartheta, w)\}, \forall \xi, \vartheta, w \in X$.
Then h is called a (α, ψ, ϕ)- generalized weakly contractive map on X.

Theorem 2.2.[3] Let h be a self map on a complete S-metric space (X,S) and h satisfies (α, ψ, ϕ)- generalized weakly contractive map. Then h have a unique fixed point in X.

Lemma 2.2.[6] Let $\{\xi_n\}$ is S_b-convergent to ξ in S_b-metric space (X, S_b) for s$\geq$1, then we obtain:
(i)$\frac{1}{2s}S_b(\vartheta,\vartheta,\xi) \leq \liminf_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq \limsup_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq 2sS_b(\vartheta,\vartheta,\xi)$ and
(ii)$\frac{1}{s^2}S_b(\xi,\xi,\vartheta) \leq liminf_{n\to\infty}S_b(\xi_n,\xi_n,\vartheta) \leq \limsup_{n\to\infty} S_b(\xi_n,\xi_n,\vartheta) \leq s^2S_b(\xi,\xi,\vartheta)$.
Lemma 2.3.[2] Let $\{\xi_n\}$ be a sequence in S_b-metric space (X,S_b) so that
$\lim_{n\to\infty} S_b(\xi_n,\xi_n,\xi_{n+1})$ = 0.
If sequence $\{\xi_n\}$ is not Cauchy, then we find an $\epsilon > 0$ and $\{m_k\}$ and $\{n_k\}$ are sequences of natural numbers with $n_k > m_k > k$ so that $S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) \geq \epsilon$, $S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) < \epsilon$ and
(i) $\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) = \epsilon$. (ii) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k-1}) = \epsilon$. (ii) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}) = \epsilon$.
In this article, we define (α,ψ,ϕ)-almost generalized weakly contractive maps in S_b-metric spaces and establish the existence and uniqueness of fixed point of maps. Also, we draw some corollaries and provide an example in support of our results.

3. MAIN RESULTS

Definition 3.1. Let (X, S_b) be an S_b-metric space for s$\geq$1. Let h be a self map of X. Then we say h be a (ψ,ϕ)-generalized weakly contractive map if $\exists$ L$\geq$0, $\psi \in \Psi$ and $\phi \in \Phi$ such that
$\psi(4s^4S_b(h\xi,h\vartheta,hw)) \leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w)$ (3.1.)
where P(ξ,ϑ,w) = max$\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}$
and Q(ξ,ϑ,w) = min$\{S_b(hw,\xi,\xi), S_b(h\xi,\vartheta,\vartheta), S_b(h\xi,w,w), S_b(h\xi,\vartheta,w)\}$
$\forall\xi,\vartheta,w \in X$.
Example 3.1. Consider (X, S_b) be a complete S_b-metric space for s=4, where $X = [0,\frac{7}{3}]$ and $S_b : X^3 \to$ R is defined by
$S_b(\xi,\vartheta,w) = \frac{1}{16}[|\xi-\vartheta| + |\vartheta-w| + |w-\xi|]^2$, $\forall\xi,\vartheta,w \in X$.
We define a self map h on X by

$$h\xi = \begin{cases} \frac{1}{8} & \text{if } \xi \in [0,2] \\ \frac{\xi}{16} - \frac{1}{32} & \text{if } \xi \in (2,\frac{7}{3}] \end{cases}.$$

Also, Consider $\phi,\psi : [0,\infty) \to [0,\infty)$ be two functions defined by ψ(t) = t and $\phi(t) = \frac{t}{4}$ for all t$\in [0,\infty)$.
Now, we verify the inequality (3.1.)
case(i) when $\xi,\vartheta,w \in$ [0,2], we have $\psi(4s^4S_b(h\xi,h\vartheta,hw))$ = 0.
Then inequality (3.1.) holds good.

case(ii) Let $\xi, \vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16} - \frac{\vartheta}{16}| + |\frac{\vartheta}{16} - \frac{w}{16}| + |\frac{w}{16} - \frac{\xi}{16}|]^2 \\ &\leq \frac{4^5}{16}[3|\frac{\xi}{16} - \frac{w}{16}|]^2 \\ &\leq \frac{9}{4}|\xi - w|^2 = \frac{1}{4} \\ &\leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\ &= \frac{3}{4}S_b(\xi, \xi, h\xi) \leq \frac{3}{4}P(\xi, \vartheta, w) \\ &= P(\xi, \vartheta, w) - \frac{1}{4}P(\xi, \vartheta, w) \\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(iii) When $\xi, \vartheta \in$[0,2] and $\in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{w}{16} - \frac{1}{32}) \\ &= \frac{4^5}{16}[|\frac{1}{8} - (\frac{w}{16} - \frac{1}{32})| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|] \\ &= \frac{4^5}{16}[2|\frac{1}{8} - \frac{w}{16} + \frac{1}{32}|]^2 \\ &= \frac{1}{4}[5 - 2w]^2 \\ &\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\ &= S_b(w, w, hw) - \frac{1}{4}S_b(w, w, hw) \\ &= P(\xi, \vartheta, w) - \frac{1}{4}P(\xi, \vartheta, w) \\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(iv) When $\vartheta, w \in [0,2]$ and $\xi \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{1}{8}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{\xi}{16} - \frac{1}{32} - \frac{1}{8}| + |0| + |\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= 4^4[\frac{5-2\xi}{32}]^2 = \frac{1}{4}[5-2\xi]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= \frac{3}{4} S_b(\xi, \xi, h\xi) \leq \frac{3}{4} P(\xi, \vartheta, w) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(v) When w$\in [0,2]$ and $\xi, \vartheta \in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[\frac{\xi - \vartheta}{16} + \frac{2\vartheta - 5}{32} + \frac{5 - 2\xi}{32}]^2 \\
&= \frac{4^5}{16}[\frac{10 - 4\vartheta}{32}]^2 \\
&\leq \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vi) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[\frac{5 - 2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5 - 2w}{32}]^2 \\
&= \frac{4^5}{16} . \frac{1}{32^2}[2(5 - 2w)]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(vii) When $\xi, w \in$[0,2] and $\vartheta \in (2, \frac{7}{3}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2 \\
&= \frac{4^5}{16} . \frac{1}{32^2}[2(5 - 2\vartheta)]^2 \\
&= \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(viii) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\ &= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\ &= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2 \\ &= \frac{1}{4}[5-2w]^2 \\ &\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\ &= S_b(w,w,hw) - \frac{1}{4}S_b(w,w,hw) \\ &= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w) \\ &= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)).\end{aligned}$$

Therefore h satisfies (ψ, ϕ) - generalized weakly contractive map.

Theorem 3.1. Suppose h be a self map on a complete symmetric S_b-metric space (X, S_b) for s$\geq$1. If h be a (ψ, ϕ) - generalized weakly contraction map, then h has a unique fixed point in X.

Proof. Let $\xi_0 \in X$ and define a sequence $\{\xi_n\}$ in X by $\xi_n = h\xi_{n-1}$, for n =1,2,3...
Suppose $\xi_{n-1} = \xi_n$ to some n, then h has a fixed point ξ_{n-1}.
Now, we suppose that $\xi_{n-1} \neq \xi_n$, $\forall\, n \in$ N.
By choosing $\xi = \vartheta = \xi_{n-2}, w = \xi_{n-1}$ in (3.1.), we obtain

$$\begin{aligned}&\psi(S_b(\xi_{n-1},\xi_{n-1},\xi_n)) \leq \psi(4s^4 S_b(h\xi_{n-2}, h\xi_{n-2}, h\xi_{n-1})) \\ &\leq \psi(P(\xi_{n-2},\xi_{n-2},\xi_{n-1})) - \phi(P(\xi_{n-2},\xi_{n-2},\xi_{n-1})) + L.Q(\xi_{n-2},\xi_{n-2},\xi_{n-1}) \qquad (3.2.)\end{aligned}$$

where

$$\begin{aligned}&P(\xi_{n-2},\xi_{n-2},\xi_{n-1}) = \\ &\max\{S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), S_b(\xi_{n-2},\xi_{n-2},h\xi_{n-2}), S_b(\xi_{n-2},\xi_{n-2},h\xi_{n-2}), \\ &\quad S_b(\xi_{n-1},\xi_{n-1},h\xi_{n-1}), \tfrac{1}{4s^2}[S_b(h\xi_{n-2},h\xi_{n-2},h\xi_{n-1}) + \\ &\quad S_b(h\xi_{n-2},h\xi_{n-2},\xi_{n-2})S_b(h\xi_{n-2},h\xi_{n-2},\xi_{n-1})S_b(h\xi_{n-1},h\xi_{n-1},\xi_{n-2})]\}) \\ &= \max\{S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), \\ &\quad S_b(\xi_{n-1},\xi_{n-1},\xi_n), \tfrac{1}{4s^2}[S_b(\xi_{n-1},\xi_{n-1},\xi_n) + \\ &\quad S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2})S_b(\xi_{n-1},\xi_{n-1},\xi_{n-1})S_b(\xi_n,\xi_n,\xi_{n-2})]\}) \\ &= \max\{S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), S_b(\xi_{n-1},\xi_{n-1},\xi_n), \tfrac{1}{4s^2}S_b(\xi_{n-1},\xi_{n-1},\xi_n)\} \\ &= max\{S_b(\xi_{n-2},\xi_{n-2},\xi_{n-1}), S_b(\xi_{n-1},\xi_{n-1},\xi_n)\} \qquad (3.3.)\end{aligned}$$

and

$Q(\xi_{n-2},\xi_{n-2},\xi_{n-1})$= min$\{S_b(h\xi_{n-1},\xi_{n-2},\xi_{n-2}), S_b(h\xi_{n-2},\xi_{n-2},\xi_{n-2}),$

$$S_b(h\xi_{n-2}, \xi_{n-1}, \xi_{n-1}), S_b(h\xi_{n-2}, \xi_{n-2}, \xi_{n-1})\}$$
$$=\min\{S_b(\xi_n, \xi_{n-2}, \xi_{n-2}), S_b(\xi_{n-1}, \xi_{n-2}, \xi_{n-2}),$$
$$S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-1}), S_b(\xi_{n-1}, \xi_{n-2}, \xi_{n-1})\}$$
$$= 0. \qquad (3.4.)$$

If $S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)$ is the maximum in (3.3.) and using (3.4.) and (3.2.), we get
$\psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)) \leq \psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)) - \phi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n))$.
This implies $\phi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n))$ = 0. Therefore, $\xi_{n-1} = \xi_n$, is a contradiction to our assumption. Thus,

$$\psi(S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)) \leq$$
$$\psi(S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})) - \phi(S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})). \qquad (3.5.)$$
$$< \psi(S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})).$$

By the definition of ψ, we have
$S_b(\xi_{n-1}, \xi_{n-1}, \xi_n) < S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1})$.
Thus, $\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)\}$ be a positive real of strictly decreasing sequence.
Then we find a r≥0 so that $\lim_{n\to\infty} S_b(\xi_{n-1}, \xi_{n-1}, \xi_n)$ = r.
Taking n$\to \infty$ in (3.5.), we obtain
$\psi(r) \leq \psi(r) - \phi(r)$. This implies $\phi(r) = 0$. Hence r = 0. Thus,

$$\lim_{n\to\infty} S_b(\xi_{n-1}, \xi_{n-1}, \xi_n) = 0. \qquad (3.6.)$$

By choosing $\xi = \vartheta = \xi_{n-1}, w = \xi_{n-2}$ in (3.1.), we get

$$\psi(S_b(\xi_n, \xi_n, \xi_{n-1})) \leq \psi(4s^4 S_b(h\xi_{n-1}, h\xi_{n-1}, h\xi_{n-2}))$$
$$\leq \psi(P(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})) - \phi(P(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})) + L.Q(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}) \qquad (3.7.)$$

where

$$P(\xi_{n-1}, \xi_{n-1}, \xi_{n-2})$$
$$= \max\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}), S_b(\xi_{n-1}, \xi_{n-1}, h\xi_{n-1}), S_b(\xi_{n-1}, \xi_{n-1}, h\xi_{n-1}),$$
$$S_b(\xi_{n-2}, \xi_{n-2}, h\xi_{n-2}), \tfrac{1}{4s^2}[S_b(h\xi_{n-1}, h\xi_{n-1}, h\xi_{n-2}) +$$
$$S_b(h\xi_{n-1}, h\xi_{n-1}, \xi_{n-1}) S_b(h\xi_{n-1}, h\xi_{n-1}, \xi_{n-2}) S_b(h\xi_{n-2}, h\xi_{n-2}, \xi_{n-1})]\}$$
$$=\max\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}), S_b(\xi_{n-1}, \xi_{n-1}, \xi_n), S_b(\xi_{n-1}, \xi_{n-1}, \xi_n),$$
$$S_b(\xi_{n-2}, \xi_{n-2}, \xi_{n-1}), \tfrac{1}{4s^2}[S_b(\xi_n, \xi_n, \xi_{n-1}) +$$
$$S_b(\xi_n, \xi_n, \xi_{n-1}) S_b(\xi_n, \xi_n, \xi_{n-2}) S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-1})]\}$$
$$=\max\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}), S_b(\xi_n, \xi_n, \xi_{n-1}), \tfrac{1}{4s^2} S_b(\xi_n, \xi_n, \xi_{n-1})\}$$
$$= max\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}), S_b(\xi_n, \xi_n, \xi_{n-1})\} \qquad (3.8.)$$

and

$$Q(\xi_{n-1}, \xi_{n-1}, \xi_{n-2}) = \min\{S_b(h\xi_{n-2}, \xi_{n-1}, \xi_{n-1}), S_b(h\xi_{n-1}, \xi_{n-1}, \xi_{n-1}),$$
$$S_b(h\xi_{n-1}, \xi_{n-2}, \xi_{n-2}), S_b(h\xi_{n-1}, \xi_{n-1}, \xi_{n-2})\}$$
$$= \min\{S_b(\xi_{n-1}, \xi_{n-1}, \xi_{n-1}), S_b(\xi_n, \xi_{n-1}, \xi_{n-1}),$$
$$S_b(\xi_n, \xi_{n-2}, \xi_{n-2}), S_b(\xi_n, \xi_{n-1}, \xi_{n-2})\}$$
$$= 0. \qquad (3.9.)$$

If $S_b(\xi_n, \xi_n, \xi_{n-1})$ is maximum in (3.8.) and using (3.7.) and (3.9.), we get

$\psi(S_b(\xi_n,\xi_n,\xi_{n-1})) \leq \psi(S_b(\xi_n,\xi_n,\xi_{n-1})) - \phi(S_b(\xi_n,\xi_n,\xi_{n-1}))$+L.0
This implies $\phi(S_b(\xi_n,\xi_n,\xi_{n-1}))$ = 0. Hence, $\xi_n = \xi_{n-1}$, is a contradiction to our assumption.
Thus

$$\psi(S_b(\xi_n,\xi_n,\xi_{n-1})) \leq \psi(S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2})) - \phi(S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2})) \quad (3.10.)$$
$$\leq \psi(S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2}))$$

From the definition of ψ, we obtain
$S_b(\xi_n,\xi_n,\xi_{n-1}) < S_b(\xi_{n-1},\xi_{n-1},\xi_{n-2})$.
Thus, $\{S_b(\xi_n,\xi_n,\xi_{n-1})\}$ be a positive reals of strictly decreasing sequence.
Hence, we can find r ≥ 0 so that
$\lim_{n\to\infty} S_b(\xi_n,\xi_n,\xi_{n-1})$ = r.
Taking $n \to \infty$ in (3.10.), we obtain
$\psi(r) \leq \psi(r) - \phi(r)$. This implies $\phi(r)$ = 0. Therefore r = 0. Thus,
$\lim_{n\to\infty} S_b(\xi_n,\xi_n,\xi_{n-1})$ = 0.
Now we verify that $\{\xi_n\}$ is a S_b-cauchy sequence in X.
Suppose that sequence $\{\xi_n\}$ is not a S_b- cauchy, $\exists\ \epsilon > 0$ and monotone increasing sequence of real numbers $m(\kappa)$ and $n(\kappa)$ with $n(\kappa) > m(\kappa) > \kappa$ so that $S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{n(\kappa)-1}) \geq \epsilon$ and $S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{n(\kappa)-2}) < \epsilon$. (3.11.)
Now from (3.1.),(3.7) and (3.11.), we have

$$\psi(4s^4\epsilon) \leq \psi(4s^4 S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{n(\kappa)-1})) =$$
$$\psi(4s^4 S_b(h\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2},h\xi_{n(\kappa)-2}))$$
$$\leq \psi(P(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2})) - \phi(P(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2}))$$
$$+ L.Q(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2})$$

where

$$P(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2})$$
$$= \max\{S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2}), S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2}),$$
$$S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2}), S_b(\xi_{n(\kappa)-2},\xi_{n(\kappa)-2},h\xi_{n(\kappa)-2}),$$
$$\frac{1}{4s^2}[S_b(h\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2},h\xi_{n(\kappa)-2})$$
$$+ S_b(h\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2},\xi_{m(\kappa)-2})S_b(h\xi_{m(\kappa)-2},h\xi_{m(\kappa)-2},$$
$$\xi_{n(\kappa)-2})S_b(h\xi_{n(\kappa)-2},h\xi_{n(\kappa)-2},\xi_{m(\kappa)-2})]\})$$
$$= \max\{S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2}), S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{m(\kappa)-1}),$$
$$S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{m(\kappa)-1}), S_b(\xi_{n(\kappa)-2},\xi_{n(\kappa)-2},\xi_{n(\kappa)-1}), \frac{1}{4s^2}[S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{n(\kappa)-1})$$
$$+ S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{m(\kappa)-2})S_b(\xi_{m(\kappa)-1},$$
$$\xi_{m(\kappa)-1},\xi_{n(\kappa)-2})S_b(\xi_{n(\kappa)-1},\xi_{n(\kappa)-1},\xi_{m(\kappa)-2})]\})$$

As $\kappa \to \infty$

$$lim_{n\to\infty} A(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2}) = \max\{S_b(\xi_{m(\kappa)-2},\xi_{m(\kappa)-2},\xi_{n(\kappa)-2}),$$
$$\frac{1}{4s^2} S_b(\xi_{m(\kappa)-1},\xi_{m(\kappa)-1},\xi_{n(\kappa)-1})\}.$$

and
$Q(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}) =$
$\min\{S_b(h\xi_{n(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}), S_b(h\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}),$
$S_b(h\xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}, \xi_{n(\kappa)-2}), S_b(h\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})\}.$
$= 0.$
If $\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})$ is maximum,
$\psi(4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) \leq$
$\psi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) - \phi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}))$
This implies
$\psi(4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) < \psi(\frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}))$
From the property of ψ, we have
$4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}) < \frac{1}{4s^2}S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})$
This gives rise to
$4s^4 < \frac{1}{4s^2} \Rightarrow 16s^6 < 1$, a contradiction as s ≥ 1.
Therefore, we have
$\psi(4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1})) \leq$
$\psi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2})) - \phi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}))$
$< \psi(S_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{n(\kappa)-2}))$
Now using lemma(2.1), we have
$4s^4 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-1}) \leq$
$2sS_b(\xi_{m(\kappa)-2}, \xi_{m(\kappa)-2}, \xi_{m(\kappa)-1}) + s^2 S_b(\xi_{m(\kappa)-1}, \xi_{m(\kappa)-1}, \xi_{n(\kappa)-2}).$
Letting $\kappa \to \infty$, we get
$4s^4\epsilon \leq s^2\epsilon$, a contradiction as s$\geq$1.
Hence $\{\xi_n\}$ be a S_b-Cauchy sequence of complete space X, $\exists \tau \in X$ so that $\lim_{n\to\infty} \xi_n = \tau$.
Now we show that $h\tau = \tau$. Suppose that $h\tau \neq \tau$. Then by lemma (2.2.), we have
$\frac{1}{2s}S_b(f\tau, f\tau, \tau) \leq \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
This implies
$\frac{4s^4}{2s}S_b(f\tau, f\tau, \tau) \leq 4s^4 \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
$\leq 4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
Thus
$2s^3 S_b(h\tau, h\tau, \tau) \leq 4s^4 \liminf_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
$\leq 4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n)$
From the property of ψ, we have
$\psi(2s^3 S_b(h\tau, h\tau, \tau)) \leq \psi(4s^4 \limsup_{n\to\infty} S_b(h\tau, h\tau, h\xi_n))$
$\leq$
$\psi(\limsup_{n\to\infty} P(\tau, \tau, \xi_n)) - \phi(\limsup_{n\to\infty} P(\tau, \tau, \xi_n)) + L(\limsup_{n\to\infty} Q(\tau, \tau, \xi_n))$
Now

$P(\tau,\tau,\xi_n)$ = max$\{S_b(\tau,\tau,\xi_n), S_b(\tau,\tau,h\tau), S_b(\tau,\tau,h\tau), S_b(\xi_n,\xi_n,h\xi_n),$
$\frac{1}{4s^2}[S_b(h\tau,h\tau,h\xi_n)+S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,\xi_n)S_b(h\xi_n,h\xi_n,\tau)]\}$
= max $\{S_b(\tau,\tau,h\tau), \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)\}$
$Q(\tau,\tau,\xi_n) = min\{S_b(h\xi_n,\tau,\tau), S_b(h\tau,\tau,\tau), S_b(h\tau,\xi_n,\xi_n), S_b(h\tau,\tau,\xi_n)\}$
= 0
If $\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)$ is maximum
$\psi(2s^3S_b(h\tau,h\tau,\tau)) \leq \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) - \phi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) + L.0$
$< \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau))$
From the property of ψ, we have
$2s^3S_b(h\tau,h\tau,\tau) < \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)$
this implies
$8s^5 < 1$, a contradiction. Therefore
$\psi(2s^3S_b(h\tau,h\tau,\tau)) \leq \psi(S_b(\tau,\tau,h\tau)) - \phi(S_b(\tau,\tau,h\tau)) + L.0$
$\Rightarrow \psi(2s^3S_b(h\tau,h\tau,\tau)) < \psi(S_b(\tau,\tau,h\tau)).$ (3.12.)
If $\tau \neq h\tau$, in (3.12.), we have
$2s^3S_b(h\tau,h\tau,\tau) < S_b(\tau,\tau,h\tau) \leq sS_b(h\tau,h\tau,\tau)$
which implies
$2s^2 < 1$, is a contradiction. Therefore, $h\tau = \tau$.
Now, we show that τ is unique.
Let τ and j be two distinct fixed points of h.
Now, consider
$\psi(S_b(\tau,\tau,j)) = \psi(S_b(h\tau,h\tau,hj))$
$\leq \psi(4s^4S_b(h\tau,h\tau,hj))$ (3.13.)
$\leq \psi(P(\tau,j,j)) - \phi(P(\tau,j,j)) + L.Q(\tau,j,j)$
where
$P(\tau,j,j)$ = max$\{S_b(\tau,j,j), S_b(\tau,\tau,h\tau), S_b(j,j,hj), S_b(j,j,hj),$
$\frac{1}{4s^4}[S_b(h\tau,hj,hj)+S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,j)S_b(hj,hj,j)]\}$
$= \{S_b(\tau,j,j), \frac{1}{4s}S_b(\tau,j,j)\} = S_b(\tau,j,j)$ (3.14.)
and $Q(\tau,j,j) = min\{S_b(fj,\tau,\tau), S_b(f\tau,j,j), S_b(f\tau,f\tau,j), S_b(fj,fj,j)\}$
$= 0$ (3.15.)
From (3.13.),(3.14.) and (3.15.) we get
$\psi(\frac{1}{4s^4}S_b(\tau,j,j)) \leq \psi(S_b(\tau,j,j)) - \phi(S_b(\tau,j,j)) + L.0$
$< \psi(S_b(\tau,j,j)).$
From the property of ψ, we have $4s^4 < 1$, a contradiction.
There fore, we get $S_b(\tau,j,j) = 0$
Hence τ=j. Hence τ is the unique fixed point of h.

In the Theorem (3.1.), if we substitue L=0, we get the following.

Corollary 3.1. Let h be a self map of X and here X is an S_b-metric space. Suppose $\exists$ $\phi \in \Phi$ and $\psi \in \Psi$ so that $\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w))$
where $P(\xi,\vartheta,w) = \max\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}.$
$\forall \xi,\vartheta,w \in X$. Then h contains unique fixed point in X.

If ψ is the identity map in the Corollary (3.1.), we get a Corollary as follows.
Corollary 3.2. Let h be a self map of X and here X is an S_b-metric space. Suppose there exists $\phi \in \Phi$ so that $4s^4 S_b(h\xi,h\vartheta,hw) \leq P(\xi,\vartheta,w) - \phi(P(\xi,\vartheta,w))$
where $P(\xi,\vartheta,w) = \max\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}.$
$\forall\ \xi,\vartheta,w \in X$. Then h contains unique fixed point in X.

If we substitute P(ξ,ϑ,w) = P*(ξ,ϑ,w) in the Theorem (3.1.), we obtain the following.
Corollary 3.3. Let h be a self map of X and here X is an S-metric space. Suppose $\exists$ $\phi \in \Phi$ and $\psi \in \Psi$ so that $\psi(4s^4 S_b(h\xi,h\vartheta,hw)) \leq \psi(P^*(\xi,\vartheta,w)) - \phi(P^*(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w)$
where P*(ξ,ϑ,w) = $\max\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{S_b(\xi,\xi,h\xi)S_b(\vartheta,\vartheta,h\vartheta)}{1+S_b(\xi,\xi,h\xi)+S_b(\xi,\vartheta,w)}, \frac{S_b(\xi,\xi,h\xi)S_b(w,w,hw)}{1+S_b(w,w,hw)+S_b(\xi,\vartheta,w)},$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}.$
and Q(ξ,ϑ,w) = $\min\{S_b(hw,\xi,\xi), S_b(h\xi,\vartheta,\vartheta), S_b(h\xi,w,w), S_b(h\xi,\vartheta,w)\}$
$\forall \xi,\vartheta,w \in X$. Then h contains unique fixed point in X.

In Theorem (3.1.), if we put s=1, we get the following.
Corollary 3.4. Let h be a self map of X and here X is an S-metric space. Suppose that $\exists$ L$\geq 0, \phi \in \Phi$ and $\psi \in \Psi$ so that
$\psi(S(h\xi,h\vartheta,hw)) \leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w)$
where $P(\xi,\vartheta,w) = \max\{S(\xi,\vartheta,w), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), S(w,w,hw),$
$\frac{1}{2}[S(h\xi,h\vartheta,hw) + S(h\xi,h\xi,\xi)S(h\xi,h\xi,w)S(hw,hw,\vartheta)]\}$
and $Q(\xi,\vartheta,w) = \min\{S(hw,\xi,\xi), S(h\xi,\vartheta,\vartheta), S(h\xi,w,w), S(h\xi,\vartheta,w)\}$
$\forall \xi,\vartheta,w \in X$. Then h contains unique fixed point in X.

Example 3.2. Consider $X = [0, \frac{12}{5}]$ and define $S : X^3 \to$ R by
$S_b(\xi,\vartheta,w) = \frac{1}{16}[|\xi-\vartheta|+|\vartheta-w|+|w-\xi|]^2, \forall \xi,\vartheta,w \in X$. Then (X, S_b) is a complete S_b-metric space for s=4.
We define a self map h on X by

$$h\xi = \begin{cases} \frac{1}{8} & \text{if } \xi \in [0,2] \\ \frac{\xi}{16} - \frac{1}{32} & \text{if } \xi \in (2, \frac{12}{5}] \end{cases}.$$

Also, Consider $\phi, \psi : [0,\infty) \rightarrow [0,\infty)$ be two functions defined by ψ(t) = t and $\phi(t) = \frac{t}{3}$, for any t $\in [0,\infty)$.
Now, we validate the inequality (3.1.).
case(i) when $\xi, \vartheta, w \in$ [0,2], we have $\psi(4s^4 S_b(h\xi, h\vartheta, hw))$ = 0.
Then inequality (3.1.) holds good.
case(ii) Let $\xi, \vartheta, w \in (2, \frac{7}{3}]$. Suppose that $\xi > \vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16} - \frac{\vartheta}{16}| + |\frac{\vartheta}{16} - \frac{w}{16}| + |\frac{w}{16} - \frac{\xi}{16}|]^2 \\
&\leq \frac{4^5}{16}[3|\frac{\xi}{16} - \frac{w}{16}|]^2 \\
&\leq \frac{9}{4}|\xi - w|^2 = \frac{9}{25} \\
&\leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(\xi, \xi, h\xi) - \frac{1}{3}S_b(\xi, \xi, h\xi) \\
&= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(iii) When $\xi, \vartheta \in$[0,2] and w$\in (2, \frac{12}{5}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{w}{16} - \frac{1}{32})| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - \frac{w}{16} + \frac{1}{32}|]^2 \\
&= \frac{1}{4}[5 - 2w]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

case(iv) When $\vartheta, w \in [0,2]$ and $\xi \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{1}{8}, \frac{1}{8}) \\ &= \frac{4^5}{16}[|\frac{\xi}{16} - \frac{1}{32} - \frac{1}{8}| + |0| + |\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\ &= \frac{4^5}{16}[2|\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\ &= 4^4[\frac{5-2\xi}{32}]^2 = \frac{1}{4}[5-2\xi]^2 \\ &\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\ &= \frac{2}{3} S_b(\xi, \xi, h\xi) \leq \frac{2}{3} P(\xi, \vartheta, w) \\ &= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(v) When w$\in [0,2]$ and $\xi, \vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\ &= \frac{4^5}{16}[\frac{\xi - \vartheta}{16} + \frac{2\vartheta - 5}{32} + \frac{5 - 2\xi}{32}]^2 \\ &= \frac{4^5}{16}[\frac{10 - 4\vartheta}{32}]^2 \\ &\leq \frac{1}{4}[5 - 2\vartheta]^2 \\ &\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\ &= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\ &= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(vi) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32})\\ &= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2\\ &= \frac{4^5}{16}[\frac{5-2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5-2w}{32}]^2\\ &= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2\\ &\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288}\\ &= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw)\\ &= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w)\\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(vii) When $\xi, w \in$[0,2] and $\vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8})\\ &= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2\\ &= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2\vartheta)]^2\\ &= \frac{1}{4}[5-2\vartheta]^2\\ &\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288}\\ &= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw)\\ &= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w)\\ &= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).\end{aligned}$$

case(viii) When $\xi \in$[0,2] and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5 - 2w)]^2 \\
&= \frac{1}{4}[5 - 2w]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)).
\end{aligned}$$

Hence the conditions of Theorem (3.1.) are satisfied by h and also $\frac{1}{8}$ is the unique fixed point of h.

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Int. J. Anal. Appl. (2023), 21:8

International Journal of Analysis and Applications

Some Invariant Point Results Using Simulation Function

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Abstract. Through this article, we establish an invariant point theorem by defining generalized Z_s-contractions in relation to the simulation function in S-metric space. In this article, we generalized the results of Nihal Tas, Nihal Yilmaz Ozgur and N.Mlaiki. In addition to that, we bestow an example which supports our results.

1. Introduction

Fixed point is also known as an invariant point. Banach principle of contraction [2] on metric space plays very important role in the field of invariant point theory and non linear analysis. In 1922, Stefan Banach initiated the concept of contraction and established well known Banach contraction theorem. In the year 2006, B Sims and Mustafa [9], established theory on G-metric spaces, that is an extension of metric spaces and established some properties. Later, A.Aliouche, S.Sedghi and N.Shobe [13] initiated S-metric spaces, it is a generalization of G-metric spaces in the year 2012. In 2014, S.Radojevic, N.V.Dung and N.T.Hieu [4] proved by examples that S-metric space is not a generalization of G-metric space and vice versa. Invariant points of various contractive maps on S-metric spaces were studied in [[1], [3], [6]- [8], [11]]. In 2015, F.Khajasteh, Satish Shukla and S.Radenovic [5] introduced simulation function and the concept of Z-contration in relation to simulation function and proved an invariant point theorem which generalizes the Banach Contraction principle. Very recently, Murat Olgun, O.Bicer and T.Alyildiz [10] defined generalized Z-contraction in relation to the simulation function and proved an invariant point theorem.

Received: Dec. 3, 2022.

2020 *Mathematics Subject Classification.* 54H25, 47H09, 47H10.

Key words and phrases. Simulation function; Z-contraction; Fixed point; S-metric space.

https://doi.org/10.28924/2291-8639-21-2023-8

ISSN: 2291-8639

In the year 2019, Nihal Tas, Nihal Ylimaz Ozgur and Nabil Mlaiki [8] proved an invariant point theorem by employing the collection of simulation mappings on S-metric spaces. In this article, we generalized the results of Nihal Tas , Nihal Yilmaz Ozgur and N.Mlaiki.

2. Preliminaries

Definition 2.1. *[13] Let $X \neq \emptyset$, then a mapping $S{:}X^3 \to [0, \infty)$ is said to be an S-metric on X if:*
(S1) $S(\xi, \vartheta, w) > 0$ for all $\xi, \vartheta, w \in X$ with $\xi \neq \vartheta \neq w$.
(S2) $S(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$.
(S3) $S(\xi, \vartheta, w) \leq [S(\xi, \xi, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$
$\forall \xi, \vartheta, w, a \in X$. Then we call (X, S) is an S-metric space.

Example 2.1. *[13] Define $S{:}X^3 \to [0, \infty)$ by $S(\xi, \vartheta, w) = d(\xi, \vartheta) + d(\xi, w) + d(\vartheta, w)$ for any $\xi, \vartheta, w \in X$, where (X, d) be a metric space. Then (X,S) is an S-metric space.*

Example 2.2. *[4] Suppose X=R, Collection of all real numbers and let $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X,S) is an S-metric space.*

Example 2.3. *[12] Suppose X=R, Collection of all real numbers and let $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X,S) is an S-metric space.*

Example 2.4. *Suppose X=[0,1] and $S{:}X^3 \to [0, \infty)$ be defined by*

$$S(\xi, \vartheta, w) = \begin{cases} 0 & if\, \xi = \vartheta = w \\ max\{\xi, \vartheta, w\} & otherwise \end{cases}.$$

Then (X,S) is an S-metric space.

Lemma 2.1. *[13] In the S-metric space, we observe $S(\xi, \xi, \vartheta) = S(\vartheta, \vartheta, \xi)$.*

Lemma 2.2. *[4] In the S-metric space, we observe*
(i) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(\vartheta, \vartheta, w)$ and
(ii) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(w, w, \vartheta)$

Definition 2.2. *[13] Let (X,S) be a S-metric space. We have:*
(i) If $S(\xi_n, \xi_n, \xi) \to 0$ as $n \to \infty$. ,then we say sequence $\{\xi_n\} \in X$ converges to $\xi \in X$. i.e., for every $\epsilon > 0$, it can be found a natural number n_0 so that to each $n \geq n_0$, $S(\xi_n, \xi_n, \xi) < \epsilon$ and we indicate it by $\lim_{n\to\infty} \xi_n = \xi$.
(ii) a sequence $\{\xi_n\} \in X$ is known as Cauchy sequence if to each $\epsilon > 0$, it can be found $n_0 \in N$ so that $S(\xi_n, \xi_n, \xi_m) < \epsilon$ for every $n,m \geq n_0$.
(iii)If each Cauchy sequence of X is convergent, then say X is complete.

Definition 2.3. *[13] A self map h is defined on S-metric space (X,S) is known as an S-contraction if we get a constant $0 \leq \tau < 1$ so that*
$S(h(\xi), h(\xi), h(\vartheta)) \leq \tau S(\xi, \xi, \vartheta)$ for all $\xi, \vartheta \in X$.

Definition 2.4. *[5] We say that a mapping $\gamma : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a simulation mapping if:*
($\gamma 1$) $\gamma(0,0) = 0$
($\gamma 2$) $\gamma(p,q) < q - p$ for p,q > 0
($\gamma 3$) If $\{p_n\}, \{q_n\}$ are sequences of (0,∞) so that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then $\lim_{n\to\infty} sup\ \gamma(p_n, q_n) < 0$.

We indicate Z as the collection of all simulation mappings. For example, $\gamma(p,q) = \tau q - p$ for $0 \leq \tau < 1$ belongning to Z.

Definition 2.5. *[5] Let h be a self map on a metric space (X,d) and $\gamma \in Z$. Then h is known as a Z-contraction in relation to γ if:*

$$\gamma(d(h\xi, h\vartheta), d(\xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in X.$$

By considering the Definition 2.5. It is concluded that each Banach contraction becomes Z-contraction in relation to $\gamma(p,q) = \tau q - p$ with $0 \leq \tau < 1$. Further, it can be established from the definition of the simulation mapping that $\gamma(p,q) < 0$ for each p $\geq q > 0$. Hence, assume that h is a Z-contraction in relation to $\gamma \in$ Z then

$$d(h\xi, h\vartheta) < d(\xi, \vartheta) \text{ for all distinct } \xi, \vartheta \in X.$$

Theorem 2.1. *[5] In complete metric space (X,d), each Z-contraction has a unique invariant point and furthermore the invariant point is the limit of every Picard's sequence.*

3. Main Results

Definition 3.1. *[13] Let h be a self map on an S-metric space X and $\gamma \in Z$. We say that h is a contraction if we find a constant $0 \leq L < 1$ such that*

$$S(h\xi, h\xi, h\vartheta) \leq LS(\xi, \xi, \vartheta) \text{ for all } \xi, \vartheta \in X.$$

Nihal Tas, N.Y.Ozgur and Nabil Mlaiki [8] defined the Z_s-contraction as follows.

Definition 3.2. *[8] Let h be a self map on an S-metric space (X, S) and $\gamma \in Z$. Then h is said to be a Z_s-contraction in relation to γ if*

$$\gamma(S(h\xi, h\xi, h\vartheta), S(\xi, \xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in X$$

Nihal Tas, N.Y.Ozgur and Nabil Mlaiki [8] proved the following theorem.

Theorem 3.1. *[8] Let h be a self map on an S-metric space (X, S). Then h has a unique invariant point a$\in$ X and the invariant point is the limit of the Picard sequence $\{\xi_n\}$, whenever h is a Z_s-contraction in relation to γ.*

Definition 3.3. *Let h be a self map on an S-metric space (X, S) and* $\gamma \in Z$*. Then h is said to be generalized* Z_s*-contraction in relation to* γ *if*

$$\gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) \geq 0 \; for \; all \; \xi, \vartheta \in X \tag{3.1}$$

where $M(\xi,\xi,\vartheta) = max\{S(\xi,\xi,\vartheta), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), \frac{1}{2}[S(\xi,\xi,h\vartheta) + S(\vartheta,\vartheta,h\xi)]\}$

Example 3.1. *Let h be a contraction on (X,S). If we take L∈[0,1) and γ(p,q) = Lq-p for all* $0 \leq p, q < \infty$*, then a contraction h is a* Z_s*-contraction in relation to* γ*. In fact, consider* $p = S(h\xi, h\xi, h\vartheta)$ *and* $q = M(\xi,\xi,\vartheta)$*. Since h is a contraction, we obtain :*

$$S(h\xi, h\xi, h\vartheta) \leq LS(\xi,\xi,\vartheta) \leq LM(\xi,\xi,\vartheta)$$
$$\implies LM(\xi,\xi,\vartheta) - S(h\xi,h\xi,h\vartheta) \geq 0$$
$$\implies \gamma(S(h\xi,h\xi,h\vartheta), M(\xi,\xi,\vartheta)) \geq 0.$$

for all $\xi, \vartheta \in X$*. Therefore, h is a generalized* Z_s*-contraction in relation to* γ*.*

Example 3.2. *Consider a complete S-metric space (X,S), where X = [0,1] and* $S : X^3 \to [0,\infty)$ *by* $S(\xi,\vartheta,w) = |\xi - w| + |\vartheta - w|$*. Define h:X → X by*

$$h\xi = \begin{cases} \frac{2}{5}, \text{ for } \xi \in [0, \frac{2}{3}) \\ \frac{1}{5}, \text{ for } \xi \in [\frac{2}{3}, 1) \end{cases}$$

Now we prove that h be a generalized Z_s*-contraction in relation to* γ*, where* γ *is defined by* $\gamma(p,q) = \frac{6}{7}q - p$*. Now we get*

$$\begin{aligned} S(h\xi,h\xi,h\vartheta) &\leq \frac{3}{7}[S(\xi,\xi,h\xi) + S(\vartheta,\vartheta,h\vartheta)] \\ &\leq \frac{6}{7}max\{S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta)\} \\ &\leq \frac{6}{7}M(\xi,\xi,\vartheta) \end{aligned}$$

for all $\xi, \vartheta \in X$*.*
That is, we have

$$\gamma(S(h\xi,h\xi,h\vartheta), M(\xi,\xi,\vartheta)) = \frac{6}{7}M(\xi,\xi,\vartheta) - d(h\xi,h\xi,h\vartheta) \geq 0.$$

for all $\xi, \vartheta \in X$*.*

Definition 3.4. *Let* (X, S) *be an S-metric space. Then we say that a mapping h:X → X is asymptotically regular at* $\xi \in X$ *if* $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$

By the following lemma, we can conclude that a generalized Z_s-contraction is asymptotically regular at each point of X.

Lemma 3.1. *If* $h : X \to X$ *is a generalized* Z_s*-contraction in relation to* γ*, then h is an asymptotically regular at each point* $\xi \in X$*.*

Proof. Let $\xi \in X$. If for some m∈N, we have $h^m\xi = h^{m-1}\xi$, that is, h$\vartheta = \vartheta$, where $\vartheta = h^{m-1}\xi$, then $h^n\vartheta = h^{n-1}h\vartheta = h^{n-1}\vartheta = \ldots =$ h$\vartheta = \vartheta$ for each n∈N. Therefore, we have:

$$\begin{aligned} S(h^n\xi, h^n\xi, h^{n+1}\xi) &= S(h^{n-m+1}h^{m-1}\xi, h^{n-m+1}h^{m-1}\xi, h^{n-m+2}h^{m-1}\xi) \\ &= S(h^{n-m+1}\vartheta, h^{n-m+1}\vartheta, h^{n-m+2}\vartheta) \\ &= S(\vartheta, \vartheta, \vartheta) \\ &= 0 \end{aligned}$$

Hence

$$\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$$

Now, we assume that $h^n\xi \neq h^{n+1}\xi$, for each n∈N.

From the condition(γ2) and the generalized Z_s-contraction property, we get:

$$0 \leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \tag{3.2}$$

Where

$$\begin{aligned} M(h^n\xi, h^n\xi, h^{n-1}\xi) = &\, max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, hh^n\xi), S(h^{n-1}\xi, h^{n-1}\xi, hh^{n-1}\xi), \\ &\frac{1}{2}[S(h^n\xi, h^n\xi, hh^{n-1}\xi) + S(h^{n-1}\xi, h^{n-1}\xi, hh^n\xi)]\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, h^{n+1}\xi), S(h^{n-1}\xi, h^{n-1}\xi, h^n\xi), \\ &\frac{1}{2}[S(h^n\xi, h^n\xi, h^n\xi) + S(h^{n-1}\xi, h^{n-1}\xi, h^{n+1}\xi)\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)\} \end{aligned}$$

If $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) > S(h^n\xi, h^n\xi, h^{n-1}\xi)$ then, we get
$M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)$
From equation (3.2) we have,

$$\begin{aligned} 0 &\leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)) \\ &< S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) = 0 \end{aligned}$$

which is a contradiction.
Hence $M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^n\xi, h^n\xi, h^{n-1}\xi)$.
Using generalized Z_s-contractive property, we get

$$\begin{aligned} 0 &\leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &= \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &< S(h^n\xi, h^n\xi, h^{n-1}\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) \end{aligned}$$

i.e., $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) < S(h^n\xi, h^n\xi, h^{n-1}\xi)$ for all n∈N.
Then $\{S(h^n\xi, h^n\xi, h^{n-1}\xi)\}$ is a nonnegative reals of decreasing sequence and so it should be convergent. Suppose $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = \eta \geq 0$. If $\eta > 0$, then from the condition (γ3) and the generalized Z_s-contraction property, we get

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi) \\ &= \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi) < 0 \end{aligned}$$

which is a contradiction. It should be $\eta = 0$.
Therefore $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$.
Hence, h is asymptotically regular at each point $\xi \in X$. □

Lemma 3.2. *The Picard sequence $\{\xi_n\}$ so that $h\xi_{n-1} = \xi_n$, to each n∈N the initial point $\xi_0 \in X$ is a bounded sequence, whenever h is a generalized Z_s-contraction in relation to γ.*

Proof. Consider $\{\xi_n\}$ be the Picard sequence in X with initial value ξ_0. Now we claim that $\{\xi_n\}$ is a bounded sequence.
Assume that $\{\xi_n\}$ is unbounded. Let $\xi_{n+m} \neq \xi_n$, for each m,n∈N.
Since $\{\xi_n\}$ is unbounded, we can find a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ so that $n_1 = 1$ and to each k∈N, n_{k+1} is the smallest integer so that
$S(\xi_{n_k+1}, \xi_{n_k+1}, \xi_{n_k}) > 1$ and $S(\xi_m, \xi_m, \xi_{n_k}) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$
Hence, from the lemma (2.2), we obtain

$$\begin{aligned} 1 &< S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) \\ &\leq 2S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + S(\xi_{n_k}, \xi_{n_k}, \xi_{n_{k+1}-1}) \\ &\leq 2S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_{k+1}-1}) + 1 \end{aligned}$$

Letting k→ ∞ and using lemma (3.1), we have

$$\lim_{n\to\infty} S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) = 1$$

$$\begin{aligned} &1 < S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}) \leq M(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}) \\ &= max\{S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\ &\frac{1}{2}[S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}})]\} \\ &= max\{S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}-1}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\ &\frac{1}{2}[S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}})]\} \end{aligned}$$

$$\leq max\{2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}),$$
$$S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \frac{1}{2}[S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_{k+1}})]\}$$
$$\leq max\{2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + 1, S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_{k+1}}),$$
$$S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \frac{1}{2}[1 + 2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{n_k}, \xi_{n_k}, \xi_{n_{k+1}})]\}$$

Letting n$\to \infty$, we get

$$1 \leq \lim_{k\to\infty} M(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}) \leq 1.$$

That is $\lim_{k\to\infty} M(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}) = 1$

From the condition(γ3) and the generalized Z_s-contraction property, we obtain

$$0 \leq \lim_{k\to\infty} sup\, \gamma(S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}), M(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1}))$$
$$= \lim_{k\to\infty} sup\, \gamma(S(\xi_{n_{k+1}}, \xi_{n_{k+1}}, \xi_{n_k}), S(\xi_{n_{k+1}-1}, \xi_{n_{k+1}-1}, \xi_{n_k-1})) < 0$$

which is a contradiction. Hence our assumption is wrong.

Therefore $\{\xi_n\}$ is bounded. □

Theorem 3.2. *Let h be a self map defined on complete S-metric space (X, S). Then h has a unique invariant point $a \in X$ and Picard sequence $\{\xi_n\}$ converges to the invariant element a, whenever h is a generalized Z_s-contraction in relation to γ.*

Proof. Let the Picard sequence $\{\xi_n\}$ be defined as h$\xi_{n-1} = \xi_n$, $\forall n \in$N and $\xi_0 \in X$. Now, we claim that $\{\xi_n\}$ be a cauchy sequence. To get this, Consider

$$T_n = sup\{S(\xi_i, \xi_i, \xi_j) : i, j \geq n\}.$$

Clearly $\{T_n\}$ be a nonnegative reals of decreasing sequence. Hence, we can find $\tau \geq 0$ so that $\lim_{n\to\infty} T_n = \tau$. Now we prove that $\tau = 0$. If possible suppose that $\tau > 0$. From the definition of T_n, for each k$\in$N, we can find m_k, n_k so that $k \leq n_k < m_k$ and

$$T_k - \frac{1}{k} < S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \leq T_k$$

Therefore, we get $\lim_{n\to\infty} S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) = \tau$.

From the lemma (2.2), lemma (3.1) and generalized Z_s-contraction property, we get

$$\begin{aligned} S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) &\leq S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \\ &\leq 2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{m_k}) \\ &\leq 2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + 2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \end{aligned}$$

Letting as k$\to \infty$, we have

$$\lim_{k\to\infty} S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \tau$$

$$
\begin{aligned}
&S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \leq M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \\
&= max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, h\xi_{m_k-1}), S(\xi_{n_k-1}, \xi_{n_k-1}, h\xi_{n_k-1}), \\
&\frac{1}{2}[S(\xi_{m_k-1}, \xi_{m_k-1}, h\xi_{n_k-1}) + S(\xi_{n_k-1}, \xi_{n_k-1}, h\xi_{m_k-1})]\} \\
&= max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\
&\frac{1}{2}[S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{m_k})]\} \\
&\leq max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\
&\frac{1}{2}[2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) + \\
&2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{n_k}, \xi_{n_k}, \xi_{m_k})]\}
\end{aligned}
$$

Letting k $\to \infty$, we get

$$\lim_{k\to\infty} M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \tau.$$

From the condition ($\gamma 3$) and the generalized Z_s-contraction property, we have

$$0 \leq \lim_{k\to\infty} sup\ \gamma(S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}), M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1})) < 0$$

This is a contraction, Hence, $\tau = 0$.

That is $\{\xi_n\}$ is a cauchy sequence in the complete S-metric space X, we can find $\eta \in X$ so that $\lim_{n\to\infty} \xi_n = \eta$.

Now we verify that, η is an invariant point of h.

If suppose h$\eta \neq \eta$, then $S(\eta, \eta, h\eta) = S(h\eta, h\eta, \eta) > 0$.

Now,

$$
\begin{aligned}
M(\xi_n, \xi_n, \eta) =& max\{S(\xi_n, \xi_n, \eta), S(\xi_n, \xi_n, h\xi_n), S(\eta, \eta, h\eta), \\
&\frac{1}{2}[S(\xi_n, \xi_n, h\eta) + S(\eta, \eta, h\xi_n)]\}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} M(\xi_n, \xi_n, \eta) &= max\{S(\eta, \eta, \eta), S(\eta, \eta, \eta), S(\eta, \eta, h\eta), \frac{1}{2}[S(\eta, \eta, h\eta) + S(\eta, \eta, \eta)]\} \\
&= S(\eta, \eta, h\eta)
\end{aligned}
$$

From the conditions $(\gamma 2), (\gamma 3)$ and Z_s-contraction property, we get

$$0 \leq \lim_{n\to\infty} sup\ \gamma(S(h\xi_n, h\xi_n, h\eta), M(\xi_n, \xi_n, \eta)) < 0$$

This is contradiction. Hence $S(\eta, \eta, h\eta) = 0 \implies$ h$\eta = \eta$.

Hence, η is a invariant point of h.

Now we claim that η is unique. Suppose α is an element in X such that $\alpha \neq \eta$ and h$\alpha = \alpha$.

Now,

$$\begin{aligned} M(\eta,\eta,\alpha) &= max\{S(\eta,\eta,\alpha), S(\eta,\eta,h\eta), S(\alpha,\alpha,h\alpha), \frac{1}{2}[S(\eta,\eta,h\alpha)+S(\alpha,\alpha,h\eta]\} \\ &= max\{S(\eta,\eta,\alpha), S(\eta,\eta,\eta), S(\alpha,\alpha,\alpha), \frac{1}{2}[S(\eta,\eta,\alpha)+S(\alpha,\alpha,\eta)]\} \\ &= S(\eta,\eta,\alpha) \end{aligned}$$

From the condition ($\gamma 2$) and Z_s-contraction property, we get

$$\begin{aligned} 0 &\leq \gamma(S(h\eta,h\eta,h\alpha), M(\eta,\eta,\alpha)) = \gamma(S(h\eta,h\eta,h\alpha), S(\eta,\eta,\alpha)) \\ &< S(\eta,\eta,\alpha) - S(\eta,\eta,\alpha) = 0, \end{aligned}$$

This is a contradiction. It should be $\eta = \alpha$. □

Example 3.3. *Consider a complete S-metric space (X, S), where $X = [0, \frac{1}{4}]$ and $S : X^3 \to [0,\infty)$ by $S(\xi,\vartheta,w) = |\xi - w| + |\xi - 2\vartheta + w|$. Define h: $X \to X$ by $h\xi = \frac{\xi}{1+\xi}$. From example 2.9 in [5], we have h be a Z-contraction in relation to $\gamma \in Z$, where $\gamma(p,q) = \frac{q}{q+\frac{1}{4}} - p$, for any p,q$\in [0,\infty)$ Therefore for all $\xi,\vartheta \in X$, we get*

$$\begin{aligned} 0 &\leq \gamma(S(h\xi,h\xi,h\vartheta), S(\xi,\xi,\vartheta)) \\ &= \frac{S(\xi,\xi,\vartheta)}{S(\xi,\xi,\vartheta)+\frac{1}{4}} - S(h\xi,h\xi,h\vartheta) \\ &\leq \frac{M(\xi,\xi,\vartheta)}{M(\xi,\xi,\vartheta)+\frac{1}{4}} - S(h\xi,h\xi,h\vartheta) \\ &= \gamma(S(h\xi,h\xi,h\vartheta), M(\xi,\xi,\vartheta)) \end{aligned}$$

Thus, h is generalized Z_s-contraction in relation to γ, for some $\gamma \in Z$. So, by using Theorem 3.2, h has a unique invariant point a=0.

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Mathematics and Statistics 10(6): 1334-1339, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100620

Some Fixed Point Results in Bicomplex Valued Metric Spaces

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Received August 10, 2022; Revised October 12, 2022; Accepted October 25, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Duduka Venkatesh, V.Naga Raju, "Some Fixed Point Results in Bicomplex Valued Metric Spaces," Mathematics and Statistics, Vol.10, No.6, pp. 1334-1339, 2022. DOI: 10.13189/ms.2022.100620*

***(b)**: Duduka Venkatesh, V.Naga Raju (2022). Some Fixed Point Results in Bicomplex Valued Metric Spaces. Mathematics and Statistics, 10(6), 1334-1339. DOI: 10.13189/ms.2022.100620*

Abstract Fixed points are also called as invariant points. Invariant point theorems are very essential tools in solving problems arising in different branches of mathematical analysis. In the present paper, we establish three unique common invariant point theorems using two self-mappings, four self-mappings and six self-mappings in the bicomplex valued metric space. In the first theorem, we generate a common invariant point theorem for four self-mappings by using weaker conditions such as weakly compatible, generalized contraction and (CLR_{AB}) property. Then, in the second theorem, we generate a common invariant point theorem for six self-mappings by using inclusion relation, generalized contraction, weakly compatible and commuting maps. Further, in the third theorem, we generate a common coupled invariant point for two self mappings using different contractions in the bicomplex valued metric space. The above results are the extention and generalization of the results of [11] in the Bicomplex metric space. Moreover, we provide an example which supports the results.

Keywords Bicomplex Valued Metric Space, Common Fixed Point, Coupled Fixed Point, CLR Property, Weakly Compatible Mappings

1 Introduction

The concepts of bicomplex numbers and tricomplex numbers were introduced in the year 1892 by Segre[1]. Complex valued metric spaces are introduced by Azam et al.[2], in the year 2011 and some results were studied for such spaces. Very recently, the bicomplex valued metric space was introduced by Cho et al.[5] and some fixed point results were obtained. In the year 2019, Jebril, Datta, Sarkar and Biswas [6] derived some fixed point outcomes using rational contractions in bicomplex valued metric space.

Imdad et al.[8] introduced a new notion, called CLR-property for self maps in 2012. Afterwards, by using it several mathematicians obtained some fixed point results ([3],[4],[9] and [10]). The main purpose of this work is to prove some invariant point outcomes using various contractions for four self mappings, six self mappings and coupled invariant point theorems using weakly compatibility, CLR_{AB} property and commuting maps in bicomplex valued metric spaces.

2 Preliminaries

We denote C_0 = R(Real numbers), C_1 = C(Complex numbers) and C_2 = Set of all bicomplex numbers.
Let $\varpi, \vartheta \in C_1$, then we define a partial order $\preceq$ on C_1 as:
$\varpi \preceq \vartheta \iff Re(\varpi) \leq Re(\vartheta)$ and $Im(\varpi) \leq Im(\vartheta)$.
Also $\varpi \prec \vartheta$ if $Re(\varpi) < Re(\vartheta)$ and $Im(\varpi) < Im(\vartheta)$.
Segre[1] defined the bicomplex number as:

$$\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2,$$

where $b_1, b_2, b_3, b_4 \in C_0$, and i_1, i_2 are the independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$,
we defined C_2 as:

$$C_2 = \{\zeta : \zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2, b_1, b_2, b_3, b_4 \in C_0\},$$

i.e.,

$$C_2 = \{\zeta : \zeta = \varpi + i_2\vartheta, \varpi, \vartheta \in C_1\}$$

where $\varpi = b_1 + b_2 i_1 \in C_1$ and $\vartheta = b_3 + b_4 i_1 \in C_1$.
If ζ=$\varpi + i_2\vartheta$ and $\gamma = u + i_2 v$ then $\zeta \pm \gamma = (\varpi + i_2\vartheta) \pm (u + i_2 v) = (\varpi \pm u) + i_2(\vartheta \pm v)$ and the product is $\zeta.\gamma$=$(\varpi+i_2\vartheta).(u+i_2 v)$=$(\varpi u-\vartheta v)+i_2(\varpi v+\vartheta u)$.

The norm $\|.\|$: $C_2 \to \mathbb{C}_0^+$ is
defined by
$\|\zeta\| = \|\varpi + i_2\vartheta\|$ =$\{|\varpi|^2 + |\vartheta|^2\}^{\frac{1}{2}} = (b_1^2 + b_2^2 + b_3^2 + b_4^2)^{\frac{1}{2}}$
where $\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2 = \varpi + i_2\vartheta \in C_2$

We define a partial order $\preceq_{i_2}$ On C_2 as:
For $\zeta = \varpi + i_2\vartheta$, γ = u+$i_2 v \in C_2$ then
$\zeta \preceq_{i_2} \gamma \iff$ if $\varpi \preceq$ u and $\vartheta \preceq$ v.
that is, $\zeta \preceq_{i_2} \gamma$ if :
(1) $\varpi = u, \vartheta = v$ or
(2) $\varpi \prec u, \vartheta = v$ or
(3) $\varpi = u, \vartheta \prec v$ or
(4) $\varpi \prec u, \vartheta \prec v$.

For any two bicomplex numbers $\zeta, \gamma \in C_2$:
(i) $\zeta \preceq_{i_2} \gamma \implies \|\zeta\| \leq \|\gamma\|$
(ii) $\|\zeta + \gamma\| \leq \|\zeta\| + \|\gamma\|$

Definition2.1.[5]Let Ω be a nonempty set. Then the mapping $\partial : \Omega \times \Omega \to C_2$ is said to bicomplex-valued metric on Ω if
1. $0 \preceq_{i_2} \partial(\varpi, \vartheta)$ for all $\varpi, \vartheta \in \Omega$,
2. $\partial(\varpi, \vartheta) = 0 \iff \varpi = \vartheta$,
3. $\partial(\varpi, \vartheta) = \partial(\vartheta, \varpi)$ for all $\varpi, \vartheta \in \Omega$ and
4. $\partial(\varpi, \vartheta) \preceq_{i_2} \partial(\varpi, u) + \partial(u, \vartheta)$ for all $\varpi, \vartheta, u \in \Omega$.
Here (Ω, ∂) is called the bicomplex valued metric space.

Let (Ω, ∂) be a bicomplex valued metric space for the following definitions:
Definition 2.2.[5]
(1). A sequence $\{\varpi_n\}$ in Ω is said to be converges to ϖ if for each $0 \prec_{i_2}$ r $\in C_2$ $\exists$ $n_0 \in$ N such that $\partial(\varpi_n, \varpi) \prec_{i_2}$ r, $\forall$ $n > n_0$ and we write $\lim_{n\to\infty} \varpi_n = \varpi$.
(2). A sequence $\{\varpi_n\}$ in Ω is said to be a cauchy sequence if for each $0 \prec_{i_2}$ r $\in C_2$ $\exists$ $n_0 \in$ N such that $\partial(\varpi_n, \varpi_{n+m}) \prec_{i_2}$ r, $\forall$ m,n $\in$ N and $n > n_0$.
(3.) We say that (Ω, ∂) is complete if each cauchy sequence of Ω is convergent.

Definition 2.3. We say that two maps h, k : $\Omega \to \Omega$ are commutes if hk(ϖ) = kh(ϖ) for all $\varpi \in \Omega$.

Definition 2.4. We say that two maps h, k : $\Omega \to \Omega$ are compatible if $\lim_{n\to\infty} \partial(hk\varpi_n, kh\varpi_n) = 0$ whenever sequence $\{\varpi_n\}$ in Ω satisfies $\lim_{n\to\infty} h\varpi_n$ = $\lim_{n\to\infty} k\varpi_n = \varpi$ for $\varpi \in \Omega$.

Definition 2.5. We say that two maps h, k : $\Omega \to \Omega$ are weakly compatible if hϖ = kϖ for some $\varpi \in \Omega$ implies hk(ϖ) = kh(ϖ).

Definition 2.6. Let h,k,A,B : $\Omega \to \Omega$ be four maps. We say that $\{h, A\}$ and $\{k, B\}$ are satisfy the CLR_{AB} property if we can find sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω satisfies $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = \varpi$ for some $\varpi \in$ A(Ω)$\cap$B(Ω).

Definition 2.7. Let h : $\Omega \times \Omega \to \Omega$ be a function. Then we say an element $(\varpi, \vartheta) \in \Omega \times \Omega$ is coupled invariant point of h if $h(\varpi, \vartheta) = \varpi$ and $h(\vartheta, \varpi) = \vartheta$.

Lemma 2.1.[7] We say a sequence $\{w_n\}$ in Ω is converges to a point w $\iff \lim_{n\to\infty} \|\partial(w_n, w)\| = 0$.

3 Main Results

Theorem 3.1. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and h,k,A and B are self mappings on Ω satisfying
(i) $\partial(h\varpi, k\vartheta) \preceq_{i_2} \tau_1\partial(A\varpi, B\vartheta) + \tau_2\partial(A\varpi, h\varpi) + \tau_3\partial(B\vartheta, k\vartheta), \forall \varpi, \vartheta \in \Omega$,
where τ_1,τ_2 and τ_3 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 < 1$.
(ii) $\{B, k\}$ and $\{A, h\}$ be weakly compatible,
(iii) $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property.
Then h,k,A and B have a one and only common invariant point.
Proof: Since $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property, then there exists sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω such that $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = \jmath$, for some $\jmath \in A\Omega \cap B\Omega$. Then $\jmath$ = Bη_1 = Aη_2, for some $\eta_1, \eta_2 \in \Omega$.
Now we prove that kη_1= Bη_1. For each n$\in \mathbb{N}$, we have
$\partial(h\varpi_n, k\eta_1) \preceq_{i_2} \tau_1\partial(A\varpi_n, B\eta_1) + \tau_2\partial(A\varpi_n, h\varpi_n) + \tau_3\partial(B\eta_1, k\eta_1)$
Letting n $\to \infty$, we get
$\partial(B\eta_1, k\eta_1) \preceq_{i_2} \tau_1\partial(B\eta_1, B\eta_1) + \tau_2\partial(B\eta_1, B\eta_1) + \tau_3\partial(B\eta_1, k\eta_1)$
i.e.,$\partial(B\eta_1, k\eta_1) \preceq_{i_2} \tau_3\partial(B\eta_1, k\eta_1)$
Therefore we have
$\|\partial(B\eta_1, k\eta_1)\| \leq \tau_3 \|\partial(B\eta_1, k\eta_1)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore we get $\|\partial(B\eta_1, k\eta_1)\|$ = 0. Thus B$\eta_1 = k\eta_1$.
Now we prove that A$\eta_2 = h\eta_2$. For each n$\in \mathbb{N}$, we condider
$\partial(h\eta_2, k\vartheta_n) \preceq_{i_2} \tau_1\partial(A\eta_2, B\vartheta_n) + \tau_2\partial(A\eta_2, h\eta_2) + \tau_3\partial(B\vartheta_n, k\vartheta_n)$
Letting n $\to \infty$, we get
$\partial(h\eta_2$,A$\eta_2) \preceq_{i_2} \tau_1\partial(A\eta_2, A\eta_2) + \tau_2\partial(A\eta_2, h\eta_2) + \tau_3\partial(A\eta_2, A\eta_2)$
i.e., $\partial(h\eta_2$,A$\eta_2) \preceq_{i_2} \tau_2\partial(h\eta_2$,A$\eta_2)$
Therefore we have $\|\partial(h\eta_2$,A$\eta_2)\| \leq \tau_2\|\partial(h\eta_2$,A$\eta_2)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore, we get $\|\partial(h\eta_2$,A$\eta_2)\|$ = 0. Thus $h\eta_2 = A\eta_2$.
Hence $B\eta_1$ = $k\eta_1$ = $h\eta_2$ = $A\eta_2$ = $\jmath$.
Given that $\{A, h\}$ is weakly compatible and hη_2 = Aη_2 then we get hAη_2 = Ahη_2. So, h$\jmath$=A$\jmath$.
Given that $\{B, k\}$ is weakly compatible and kη_1 = Bη_1 then we get kBη_1 = Bkη_1. So, k$\jmath$ = B$\jmath$.

Now we prove that h$\jmath$ = $\jmath$:
Consider $\partial(h\jmath,k\eta_1) \preceq_{i_2} \tau_1\partial(A\jmath,B\eta_1) + \tau_2\partial(A\jmath,h\jmath) + \tau_3\partial(B\eta_1,k\eta_1)$
i.e.,, $\partial(h\jmath,\jmath) \preceq_{i_2} \tau_1\partial(h\jmath,\jmath) + \tau_2\partial(h\jmath,h\jmath) + \tau_3\partial(\jmath,\jmath)$
i.e.,, $\partial(h\jmath,\jmath) \preceq_{i_2} \tau_1\partial(h\jmath,\jmath)$
Therefore we have $\|\partial(h\jmath,\jmath)\| \leq \tau_1 \|\partial(h\jmath,\jmath)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore, we get $\|\partial(h\jmath,\jmath)\| = 0$. Thus h$\jmath$ = $\jmath$. So, we have h$\jmath$ = $\jmath$ = A$\jmath$.
Now we prove that k$\jmath$ = $\jmath$:
Consider
$\partial(h\eta_2,k\jmath) \preceq_{i_2} \tau_1\partial(A\eta_2,B\jmath) + \tau_2\partial(A\eta_2,h\eta_2) + \tau_3\partial(B\jmath,k\jmath)$
i.e., $\partial(\jmath,k\jmath) \preceq_{i_2} \tau_1\partial(\jmath,k\jmath) + \tau_2\partial(\jmath,\jmath) + \tau_3\partial(k\jmath,k\jmath)$
Therefore we have $\|\partial(\jmath,k\jmath)\| \leq \tau_1\|\partial(\jmath,k\jmath)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.
Therefore, we get $\|\partial(\jmath,k\jmath)\| = 0$. Thus k$\jmath$ = $\jmath$. So, we have k$\jmath$ = $\jmath$ = B$\jmath$.
Hence h$\jmath$ = A$\jmath$ = $\jmath$ = k$\jmath$ = B$\jmath$.
Therefore $\jmath$ is common invariant point of A,h,k and B.
Now we prove $\jmath$ is unique:
For this, we consider δ is any other common invariant point of h,k,A and B.
Then hδ = kδ = Aδ = Bδ = δ.
Now, Consider
$\partial(\jmath,\delta) = \partial(h\jmath,k\delta) \preceq_{i_2} \tau_1\partial(A\jmath,B\delta) + \tau_2\partial(A\jmath,h\jmath) + \tau_3\partial(B\delta,k\delta)$
i.e., $\partial(\jmath,\delta) \preceq_{i_2} \tau_1\partial(\jmath,\delta) + \tau_2\partial(\jmath,\jmath) + \tau_3\partial(\delta,\delta)$
i.e., $\partial(\jmath,\delta) \preceq_{i_2} \tau_1\ \partial(\jmath,\delta)$
Therefore we have $\|\partial(\jmath,\delta)\| \leq \tau_1\|\partial(\jmath,\delta)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Hence, we get $\|\partial(\jmath,\delta)\| = 0$.Thus $\jmath = \delta$.
Hence, $\jmath$ is the one and only one common invariant point of h,k,A and B.

Example 3.1. Consider Ω = [0,1] and define $\partial\colon \Omega\times\Omega \to C_2$ by

$$\partial(\varpi,\vartheta) = \begin{cases} 0, & \text{for } \varpi = \vartheta \quad and \\ i_2 max\{\varpi,\vartheta\}, & otherwise \end{cases}$$

for all $\varpi,\vartheta \in \Omega$.
Define h,k,A and B be self maps on Ω defined as:
For $\varpi \in \Omega, h(\varpi) = \frac{\varpi}{3}, k(\varpi) = \frac{\varpi}{3}, A(\varpi) = \varpi$ and $B(\varpi) = \varpi$.
Case(i): We show that {h,A} and {k,B} satisfy CLR_{AB} property. For this , we choose $\varpi_n = \frac{1}{2n}$ and $\vartheta_n = \frac{1}{3n+1}$ for n$\in$ **N**. Clearly, $<\varpi_n>$ and $<\vartheta_n>$ are in Ω. Then $\partial(A\varpi_n,0) = \partial(\frac{1}{2n},0)$ converges to 0 as $n \to \infty$. Also, $\partial(h\varpi_n,0) = \partial(\frac{1}{6n},0)$ converges to 0 as $n \to \infty$. Similarly, we get $\partial(k\vartheta_n,0) = \partial(\frac{1}{9n+1},0) \to 0$ as $n \to \infty$. and $\partial(B\vartheta_n,0) = \partial(\frac{1}{3n+1},0) \to 0$ as $n \to \infty$.
Since A0 = 0 = B0, So, we have $0 \in A\Omega \cap B\Omega$. Therefore, we have sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω so that $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = 0$. Thus {h,A} and {k,B} satisfies CLR_{AB} property.
case(ii): we show that {h,A} and {k,B} are weakly compatible. Now, $h\varpi = A\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and hA(0) = h(0) = 0 and Ah(0) = A(0) = 0. Thus $hA(\varpi) = Ah(\varpi)$, whenever $h\varpi = A\varpi$, for all $\varpi \in \Omega$. Hence {h,A} is weakly compatible in Ω.
Also, $k\varpi = B\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and kB(0) = Bk(0). Thus, $kB(\varpi) = Bk(\varpi)$, whenever $k\varpi = B\varpi$ for all $\varpi \in \Omega$. Hence, {k,B} is weakly compatible in Ω.
case(iii): Now, $\partial(h\varpi,k\vartheta) = \partial(\frac{\varpi}{3},\frac{\vartheta}{3}) = i_2 max\{\frac{\varpi}{3},\frac{\vartheta}{3}\}$,
$\partial(A\varpi,B\vartheta) = \partial(\varpi,\vartheta) = i_2 max\{\varpi,\vartheta\}$,
$\partial(A\varpi,h\varpi) = \partial(\varpi,\frac{\varpi}{3}) = i_2\max\{\varpi,\frac{\varpi}{3}\} = i_2\varpi$,
$\partial(B\vartheta,k\vartheta) = \partial(\vartheta,\frac{\vartheta}{3}) = i_2 max\{\vartheta,\frac{\vartheta}{3}\} = i_2\vartheta$.
subcase(i) if $\varpi > \vartheta$ then
$\partial(h\varpi,k\vartheta) = i_2 max\{\frac{\varpi}{3},\frac{\vartheta}{3}\} = i_2\frac{\varpi}{3}$,
$\partial(A\varpi,B\vartheta) = i_2 max\{\varpi,\vartheta\} = i_2\varpi$,
$\partial(A\varpi,h\varpi) = i_2 max\{\varpi,\frac{\varpi}{3}\} = i_2\varpi$, $\partial(B\vartheta,k\vartheta) = i_2\vartheta$.
Now, $\partial(h\varpi,k\vartheta) = i_2\frac{\varpi}{3} \preceq_{i_2} \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$
i.e., $\partial(h\varpi,k\vartheta) \preceq_{i_2} \frac{1}{4}\partial(A\varpi,B\vartheta) + \frac{1}{4}\partial(A\varpi,h\varpi) + \frac{1}{4}\partial(B\vartheta,k\vartheta)$
By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$, Here τ_1,τ_2,τ_3 be nonnegative real numbers such that $\tau_1 + \tau_2 + \tau_3 < 1$. Hence
$d(h\varpi,k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi,B\vartheta) + \tau_2 d(A\varpi,h\varpi) + \tau_3 d(B\vartheta,k\vartheta)$.
subcase(ii) if $\varpi < \vartheta$ then
$\partial(h\varpi,k\vartheta) = i_2\max\{\frac{\varpi}{3},\frac{\vartheta}{3}\} = i_2\frac{\vartheta}{3}$,
$\partial(A\varpi,B\vartheta) = i_2 max\{\varpi,\vartheta\} = i_2\vartheta$,
$\partial(A\varpi,h\varpi) = i_2 max\{\varpi,\frac{\varpi}{3}\} = i_2\varpi$, $\partial(B\vartheta,k\vartheta) = i_2\vartheta$.
Now, $\partial(h\varpi,k\vartheta) = i_2\frac{\vartheta}{3} \preceq_{i_2} \frac{1}{4}[i_2\vartheta] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$
i.e., $\partial(h\varpi,k\vartheta) \preceq_{i_2} \frac{1}{4}\partial(A\varpi,B\vartheta) + \frac{1}{4}\partial(A\varpi,h\varpi) + \frac{1}{4}\partial(B\vartheta,k\vartheta)$
By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$,
Here τ_1,τ_2,τ_3 be nonnegative real numbers such that $\tau_1 + \tau_2 + \tau_3 < 1$. Hence
$\partial(h\varpi,k\vartheta) \preceq_{i_2} \tau_1\partial(A\varpi,B\vartheta) + \tau_2\partial(A\varpi,h\varpi) + \tau_3\partial(B\vartheta,k\vartheta)$.

Corollary 3.1. Suppose (Ω,∂) be a complete Bicomplex valued metric space and h,k and A be self mappings on Ω satisfies
(i) $\partial(hz,kw) \preceq_{i_2} \tau_1\partial(Az,Aw) + \tau_2\partial(Az,hz) + \tau_3\partial(Aw,kw)$, for all z,w $\in$ Ω, where τ_1,τ_2 and τ_3 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 < 1$.
(ii) $\{h,A\}$ and $\{k,A\}$ are weakly compatible,
(iii) $\{h,A\}$ and $\{k,A\}$ satisfy CLR_A property.
Then h,k and A have a unique common invariant point.
Proof: We can prove this results easily by substituting B = A in the Theorem 3.1.

Theorem 3.2. Suppose (Ω,∂) be a complete Bicomplex valued metric space and H,I,C,P,Q,R be the self mappings on Ω satisfies (i) H(Ω) $\supseteq$ QR(Ω) and I(Ω) $\supseteq$ CP(Ω) (ii) $\partial(CP\varpi,QR\vartheta) \preceq_{i_2} \tau_1\partial(H\varpi,I\vartheta) + \tau_2\partial(H\varpi,CP\varpi) + \tau_3\partial(I\vartheta,QR\vartheta) + \tau_4\partial(H\varpi,QR\vartheta)$ for all $\varpi,\vartheta \in \Omega$, where τ_1,τ_2,τ_3 and τ_4 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$. (iii) Suppose (QR,I) and (CP,H) be weakly compatible. (Q,R), (Q,I) (R,I),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P,I and H contains one and only one common invariant point in Ω.
Proof: Let $\varpi_0 \in \Omega$. Since $H(\Omega) \supseteq$ QR(Ω) and I(Ω) $\supseteq$ CP(Ω) then we can find a sequence $\{\varpi'_n\}$ in Ω such that
CPϖ_{2l} = I$\varpi_{2l+1} = \varpi'_{2l}$ and QRϖ_{2l+1} = H$\varpi_{2l+2} = \varpi'_{2l+1}$ for l=0,1,2,....
Consider $\partial(\varpi'_{2l},\varpi'_{2l+1}) = \partial(CP\varpi_{2l},QR\varpi_{2l+1})$
$\preceq_{i_2} \tau_1\partial(H\varpi_{2l},I\varpi_{2l+1}) + \tau_2\partial(H\varpi_{2l},CP\varpi_{2l})$
$+ \tau_3\partial(I\varpi_{2l+1},QR\varpi_{2l+1}) + \tau_4\partial(H\varpi_{2l},QR\varpi_{2l+1})$

= $\tau_1\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$
+ $\tau_4\ \partial(\varpi'_{2l-1}, \varpi'_{2l+1})$
= $\tau_1\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$
+ $\tau_4\ [\partial(\varpi'_{2l-1}, \varpi'_{2l})$+ $\partial(\varpi'_{2l}, \varpi'_{2l+1})]$
i.e., $(1 - \tau_3 - \tau_4)\ \partial(\varpi'_{2l}, \varpi'_{2l+1}) \preceq_{i_2} (\tau_1 + \tau_2 + \tau_4)\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$
i.e., $\partial(\varpi'_{2l}, \varpi'_{2l+1}) \preceq_{i_2} (\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4})\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$
Similarly, we consider
$\partial(\varpi'_{2l+1}, \varpi'_{2l+2}) = \partial(QR\varpi_{2l+1}, \text{CP}\varpi_{2l+2})$
= $\partial(CP\varpi_{2l+2}, \text{QR}\varpi_{2l+1})$
$\preceq_{i_2} \tau_1\ \partial(H\varpi_{2l+2}, \text{I}\varpi_{2l+1})$ + $\tau_2\ \partial(H\varpi_{2l+2}, \text{CP}\varpi_{2l+2})$
+ $\tau_3\ \partial(I\varpi_{2l+1}, \text{QR}\varpi_{2l+1})$ + $\tau_4\ \partial(H\varpi_{2l+2}, \text{QR}\varpi_{2l+1})$
= $\tau_1\ \partial(\varpi'_{2l+1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l+1}, \varpi'_{2l+2})$
+ $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$ + $\tau_4\ \partial(\varpi'_{2l+1}, \varpi'_{2l+1})$
i.e.,(1-τ_2) $\partial(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\tau_1 + \tau_3)\partial(\varpi'_{2l}, \varpi'_{2l+1})$
i.e.,$\partial(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\frac{\tau_1+\tau_3}{1-\tau_2})\partial(\varpi'_{2l}, \varpi'_{2l+1})$
Let us consider σ = max $\{\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4}, \frac{\tau_1+\tau_3}{1-\tau_2}\}$
then σ <1, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Now, for m,l $\in$ N and l < m, we consider
$\partial(\varpi'_l, \varpi'_m) \preceq_{i_2} \partial(\varpi'_l, \varpi'_{l+1})$ + $\partial(\varpi'_{l+1}, \varpi'_{l+2})$ +.....
+ $\partial(\varpi'_{m-1}, \varpi'_m)$
$\preceq_{i_2} (\sigma^l + \sigma^{l+1} + ... + \sigma^{m-1})\partial(\varpi'_0, \varpi'_1)$
i.e., $\partial(\varpi'_l, \varpi'_m) \preceq_{i_2} (\frac{\sigma^l}{1-\sigma})\partial(\varpi'_0, \varpi'_1)$
Therefore we obtain
$\|\partial(\varpi'_l, \varpi'_m)\| \preceq_{i_2} (\frac{\sigma^l}{1-\sigma})\|\partial(\varpi'_0, \varpi'_1)\|$
Since $\sigma < 1$, as n,m $\to \infty$, we get $\|\partial(\varpi'_l, \varpi'_m)\| \to 0$
Hence $\{\varpi'_n\}$ be a cauuchy sequence in complete space Ω, then $\exists\ \jmath \in \Omega$ such that
$\lim_{n\to\infty} CP\varpi_{2n}$ = $\lim_{n\to\infty} I\varpi_{2n+1}$ = $\lim_{n\to\infty} QR\varpi_{2n+1}$ = $\lim_{n\to\infty} P\varpi_{2n+2}$ = $\jmath$.
Since QR(Ω) $\subseteq$ H(Ω), then $\exists$ z $\in \Omega$ such that Hz = $\jmath$.
Now we consider
$\partial(CPz, \jmath) \preceq_{i_2} \partial(CPz, QR\varpi_{2n+1}) + \partial(QR\varpi_{2n+1}, \jmath)$
$\preceq_{i_2} \tau_1\partial(Hz, I\varpi_{2n+1})$ + $\tau_2\ \partial(Hz, CPz)$
+ $\tau_3\ \partial(I\varpi_{2n+1}, QR\varpi_{2n+1})$
+ $\tau_4\ \partial(Hz, QR\varpi_{2n+1}) + \partial(QR\varpi_{2n+1}, \jmath)$
Letting n $\to \infty$, we have
(CPz, $\jmath$) $\preceq_{i_2} \tau_1\partial(\jmath, \jmath) + \tau_2\partial(\jmath, CPz) + \tau_3\partial(\jmath, \eta)$
+ $\tau_4\partial(\jmath, \jmath) + \partial(\jmath, \jmath)$
Therefore we get
$\|\partial(CPz, \jmath)\| \leq \tau_2\|\partial(CPz, \jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore we get, $\|\partial(CPz, \jmath)\|$ = 0.
Hence CPz = Hz = $\jmath$.
Again Since, CP(Ω) $\subseteq$ I(Ω), so there exists w $\in \Omega$ with Iw = $\jmath$.
Now we consider,
$\partial(\jmath, QRw) = \partial(CPz, QRw)$
$\preceq_{i_2} \tau_1\partial(Hz, Iw) + \tau_2\partial(Hz, CPz) + \tau_3\partial(Iw, QRw)$
+ $\tau_4\partial(Hz, QRw)$
i.e., $\partial(\jmath, QRw) \preceq_{i_2} (\tau_3 + \tau_4)\partial(\jmath, QRw)$
i.e., $\|\partial(\jmath, QRw)\| \preceq_{i_2} (\tau_3 + \tau_4)\|\partial(\jmath, QRw)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore we get,$\|\partial(\jmath, QRw)\|$ = 0.
Hence QRw = $\jmath$ = Iw.
Thus we get CPz = Hz = QRw = Iw = $\jmath$.

Since I and QR are weakly compatible, then I(QR)w = QR(I)w implies I$\jmath$ = QR$\jmath$.
Since CP and H are weakly compatible, then (CP)Hz = H(CP)z implies CP$\jmath$ = H$\jmath$.
Now we show that CP$\jmath$ = H$\jmath$ = $\jmath$:
We now consider
$\partial(CP\jmath, \jmath) = \partial(CP\jmath, QRw)$
$\preceq_{i_2} \tau_1\partial(H\jmath, IIw) + \tau_2\partial(H\jmath, CP\jmath) + \tau_3\partial(Iw, QRw)$
+ $\tau_4\partial(H\jmath, QRw)$
= $\tau_1\partial(CP\jmath, \jmath) + \tau_2\partial(H\jmath, H\jmath) + \tau_3\partial(Iw, Iw)$
+ $\tau_4\partial(CP\jmath, \jmath)$
i.e., $\partial(CP\jmath, \jmath) \preceq_{i_2} (\tau_1 + \tau_4)\partial(CP\jmath, \jmath)$
i.e., $\|\partial(CP\jmath, \jmath)\| \leq (\tau_1 + \tau_4)\|\partial(CP\jmath, \jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(CP\jmath, \jmath)\|$ = 0.
Hence $CP\jmath = \jmath = H\jmath$.
Now, we show that $QR\jmath = \jmath$:
We now consider
$\partial(\jmath, QR\jmath) = \partial(CP\jmath, QR\jmath)$
$\preceq_{i_2} \tau_1\partial(H\jmath, I\jmath) + \tau_2\partial(H\jmath, CP\jmath) + \tau_3\partial(I\jmath, QR\jmath)$
+ $\tau_4\partial(H\eta, QR\eta)$
= $\tau_1\partial(\jmath, QR\jmath) + \tau_2\partial(H\jmath, H\jmath) + \tau_3\partial(I\jmath, I\jmath)$
+ $\tau_4\partial(\jmath, QR\jmath)$
i.e., $\partial(\jmath, QR\jmath) \preceq_{i_2} (\tau_1 + \tau_4)\partial(\jmath, QR\jmath)$
i.e., $\|\partial(\jmath, QR\jmath)\| \leq (\tau_1 + \tau_4)\|\partial(\jmath, QR\jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(\jmath, QR\jmath)\|$ = 0.
Hence, $QR\jmath = \jmath = I\jmath$.
Thus, we get $CP\jmath = H\jmath = QR\jmath = I\jmath = \jmath$.
So, $\jmath$ be a common invariant point of H,I,CP and QR.
Since we have commuting conditions of pairs, we get
$Q\jmath = Q(QR\jmath) = Q(RQ\jmath) = (QR)Q\jmath$ and
$Q\jmath = Q(H\jmath) = H(Q\jmath)$;
$R\jmath = R(H\jmath) = HR\jmath$ and
$R\jmath = R(QR\jmath) = (RQ)R\jmath = (QR)R\jmath$.
Thus Q$\jmath$ and R$\jmath$ are common invariant points of (QR,H).
Therefore, we get Q$\jmath$ = $\jmath$ = R$\jmath$ = H$\jmath$ = QR$\jmath$.
Simiarly, we can easily prove, C$\jmath$ = $\jmath$ = P$\jmath$ = I$\jmath$ = CP$\jmath$.
Thus, $\jmath$ be a common invariant point of H,I,C,P,Q and R.
Now we prove $\jmath$ is unique. Suppose γ be common invariant point of H,I,C,P,Q and R other than $\jmath$.
Now we consider,
$\partial(\jmath, \gamma) = \partial(CP\jmath, QR\gamma)$
$\preceq_{i_2} \tau_1\partial(H\jmath, I\gamma) + \tau_2\partial(H\jmath, CP\eta) + \tau_3\partial(I\gamma, QR\gamma)$
+ $\tau_4\partial(H\jmath, QR\gamma)$
i.e., $\partial(\jmath, \gamma) \preceq_{i_2} (\tau_1 + \tau_4)\partial(\jmath, \gamma)$
i.e.,$\|\partial(\jmath, \gamma)\| \preceq_{i_2} (\tau_1 + \tau_4)\|\partial(\jmath, \gamma)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(\jmath, \gamma)\|$ = 0.
Hence, we get $\jmath = \gamma$.
Thus $\jmath$ is the one and only common invariant point of H,I,C,P,Q and R.

Corollary 3.2. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and H,C,P,Q,R be the self mappings on Ω satisfies (i) $H(\Omega) \supseteq QR(\Omega)$ and $H(\Omega) \supseteq CP(\Omega)$ (ii) $\partial(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1\partial(H\varpi, H\vartheta) + \tau_2\partial(H\varpi, CP\varpi) + \tau_3\partial(H\vartheta, QR\vartheta) + \tau_4\partial(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in \Omega$, where τ_1,τ_2,τ_3 and τ_4 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$. (iii) Suppose that (QR,H) and (CP,H) be weakly compatible. (Q,R), (Q,H) (R,H),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P and H have a unique common invariant point in Ω.

Proof: This results can be prove easily by substituting I = H in the above theorem.

Theorem 3.3. Suppose (Ω, ∂) be a complete Bicomplex metric space and h,k:$\Omega \times \Omega \to \Omega$ be two functions satisfies

$\partial(h(\varpi, \jmath), k(\rho, \sigma)) \preceq_{i_2} \tau_1 \frac{\partial(\varpi,\rho)+\partial(\jmath,\sigma)}{2}$
$+ \tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\rho,\varpi)}{2} + \tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\rho,k(\rho,\sigma))}{2}$

where $\varpi, \jmath, \rho, \sigma \in \Omega$ and τ_1, τ_2 and τ_3 are nonnegative integers such that $1 > \tau_1 + \tau_2 + \tau_3$. Then h and k contains one and only common coupled invariant point in $\Omega \times \Omega$.

Proof: Consider two arbitrary elements $\varpi_0, \jmath_0 \in \Omega$. We define two sequences $\{\varpi_n\}, \{\jmath_n\}$ such that $\varpi_{2l+1} = h(\varpi_{2l}, \jmath_{2l}), \varpi_{2l+2} = k(\varpi_{2l+1}, \jmath_{2l+1})$, $\jmath_{2l+1} = h(\jmath_{2l}, \varpi_{2l}), \jmath_{2l+2} = k(\jmath_{2l+1}, \varpi_{2l+1})$, for l=0,1,2...

Now we consider,

$\partial(\varpi_{2l+1}, \varpi_{2l+2}) = \partial(h(\varpi_{2l}, \jmath_{2l}), k(\varpi_{2l+1}, \jmath_{2l+1}))$
$\preceq_{i_2} \tau_1 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\jmath_{2l},\jmath_{2l+1})}{2}$
$+ \tau_2 \frac{\partial(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+\partial(\varpi_{2l+1},\varpi_{2l})}{2}$
$+ \tau_3 \frac{\partial(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+\partial(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2}$
$= \tau_1 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\jmath_{2l},\jmath_{2l+1})}{2} + \tau_2 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\varpi_{2l+1},\varpi_{2l})}{2} +$
$\tau_3 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\varpi_{2l+1},\varpi_{2l+2})}{2}$
$= (\frac{\tau_1+2\tau_2+\tau_3}{2})\partial(\varpi_{2l}, \varpi_{2l+1}) + (\frac{\tau_1}{2})\, \partial(\jmath_{2l}, \jmath_{2l+1})$
$+ (\frac{\tau_3}{2})\partial(\varpi_{2l+1}, \varpi_{2l+2})$

i.e.,

$(2 - \tau_3)\partial(\varpi_{2l+1}, \varpi_{2l+2}) \preceq_{i_2} (\tau_1 + 2\tau_2 + \tau_3)\partial(\varpi_{2l}, \varpi_{2l+1})$
$+ (\tau_1)\, \partial(\jmath_{2l}, \jmath_{2l+1})$–(3.1)

Again, we consider

$\partial(\jmath_{2l+1}, \jmath_{2l+2}) = \partial(h(\jmath_{2l}, \varpi_{2l}), k(\jmath_{2l+1}, \varpi_{2l+1}))$
$\preceq_{i_2} \tau_1 \frac{\partial(\jmath'_{2l},\jmath_{2l+1})+\partial(\varpi_{2l},\varpi_{2l+1})}{2}$
$+ \tau_2 \frac{\partial(\jmath_{2l},h(\jmath_{2l},\varpi_{2l}))+\partial(\jmath_{2l+1},\jmath_{2l})}{2}$
$+ \tau_3 \frac{\partial(\jmath_{2l},h(\jmath_{2l},\varpi_{2l}))+\partial(\jmath_{2l+1},k(\jmath_{2l+1},\varpi_{2l+1}))}{2}$
$= \tau_1 \frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\varpi_{2l},\varpi_{2l+1})}{2} + \tau_2 \frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\jmath_{2l+1},\jmath_{2l})}{2}$
$+ \tau_3 \frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\jmath_{2l+1},\jmath_{2l+2})}{2}$
$= (\frac{\tau_1+2\tau_2+\tau_3}{2})\partial(\jmath_{2l}, \jmath_{2l+1}) + (\frac{\tau_1}{2})\, \partial(\varpi_{2l}, \varpi_{2l+1})$
$+ (\frac{\tau_3}{2})\partial(\jmath_{2l+1}, \jmath_{2l+2})$

i.e.,

$(2 - \tau_3)\partial(\jmath_{2l+1}, \jmath_{2l+2}) \preceq_{i_2} (\tau_1 + 2\tau_2 + \tau_3)\partial(\jmath_{2l}, \jmath_{2l+1})$
$+ (\tau_1)\, \partial(\varpi_{2l}, \varpi_{2l+1})$ –(3.2)

By adding the equations (3.1) and (3.2) we get

$\partial(\varpi_{2l+1}, \varpi_{2l+2}) + \partial(\jmath_{2l+1}, \jmath_{2l+2}) \preceq_{i_2} \eta[\partial(\varpi_{2l}, \varpi_{2l+1}) + \partial(\jmath_{2l}, \jmath_{2l+1})]$

where $\eta = \frac{2\tau_1+2\tau_2+\tau_3}{2-\tau_3}$ and $0 \le \eta < 1$, Since $1 > \tau_1 + \tau_2 + \tau_3$.

Similarly, we can easily show that

$\partial(\varpi_{2l+2}, \varpi_{2l+3}) + \partial(\jmath_{2l+2}, \jmath_{2l+3}) \preceq_{i_2} \eta[\partial(\varpi_{2l+1}, \varpi_{2l+2}) + \partial(\jmath_{2l+1}, \jmath_{2l+2})]$

Then, for any l $\in$ N, we get

$\partial(\varpi_{l+2}, \varpi_{l+1}) + \partial(\jmath_{l+2}, \jmath_{l+1}) \preceq_{i_2} \eta[\partial(\varpi_{l+1}, \varpi_l) + \partial(\jmath_{l+1}, \jmath_l)]$
$\preceq_{i_2} \eta^2[\partial(\varpi_l, \varpi_{l-1}) + \partial(\jmath_l, \jmath_{l-1})]$
....
$\preceq_{i_2} \eta^{l+1}[\partial(\varpi_1, \varpi_0) + \partial(\jmath_1, \jmath_0)]$

Now, we consider $m, l \in$ N and m >l , we get

$\partial(\varpi_m, \varpi_l) + \partial(\jmath_m, \jmath_l) \preceq_{i_2} [\partial(\varpi_l, \varpi_{l+1}) + \partial(\jmath_l, \jmath_{l+1})]$
$+ [\partial(\varpi_{l+1}, \varpi_m) + \partial(\jmath_{l+1}, \jmath_m)]$
$\preceq_{i_2} [\partial(\varpi_l, \varpi_{l+1}) + \partial(\jmath_l, \jmath_{l+1})] + [\partial(\varpi_{l+1}, \varpi_{l+2})$
$+ \partial(\jmath_{l+1}, \jmath_{l+2})] + ... + [\partial(\varpi_{m-1}, \varpi_m) + \partial(\jmath_{m-1}, \jmath_m)]$
$\preceq_{i_2} [\eta^l + \eta^{l+1} + \eta^{l+2} + ... + \eta^{m-1}][\partial(\varpi_1, \varpi_0) + \partial(\jmath_1, \jmath_0)]$
$\preceq_{i_2} (\frac{\eta^l}{1-\eta})\, [\partial(\varpi_1, \varpi_0) + \partial(\jmath_1, \jmath_0)]$

Since $0 \le \eta < 1$, Then $\partial(\varpi_m, \varpi_l) \to 0$ & $\partial(\jmath_m, \jmath_l) \to 0$, as l,m $\to \infty$.

Hence $\{\varpi_n\}$ and $\{\jmath_n\}$ be two cauchy sequences in X and there exists $\varpi, \jmath \in$ X such that $(\varpi_n) \to \varpi$ and $(\jmath_n) \to \jmath$ as n$\to \infty$.

Now we consider

$\partial(h(\varpi, \jmath), \varpi) \preceq_{i_2} \partial(h(\varpi, \jmath), \varpi_{2l+2}) + \partial(\varpi_{2l+2}, \varpi)$
$= \partial(h(\varpi, \jmath), k(\varpi_{2l+1}, s_{2l+1}) + \partial(\varpi_{2l+2}, \varpi)$
$\preceq_{i_2} \tau_1 \frac{\partial(\varpi,\varpi_{2l+1})+\partial(\jmath,\jmath_{2l+1})}{2} + \tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi)}{2}$
$+ \tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2} + \partial(\varpi_{2l+2}, \varpi)$
$= \tau_1 \frac{\partial(\varpi,\varpi_{2l+1})+\partial(\jmath,\jmath_{2l+1})}{2} + \tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi)}{2}$
$+ \tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi_{2l+2})}{2} + \partial(\varpi_{2l+2}, \varpi)$

Letting the limit as l $\to \infty$, then we get

$\|\partial(h(\varpi, \jmath), \varpi)\| \le (\frac{\tau_2+\tau_3}{2})\|\partial(h(\varpi, \jmath), \varpi)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|\partial(h(\varpi, \jmath), \varpi)\| = 0$.

Hence, $h(\varpi, \jmath) = \varpi$. Similarly it can easily show that $h(\jmath, \varpi) = \jmath$.

Now we consider,

$\partial(\varpi, k(\varpi, \jmath)) = \partial(h(\varpi, \jmath), k(\varpi, \jmath))$
$\preceq_{i_2} \tau_1 \frac{\partial(\varpi,\varpi)+d(\jmath,\jmath)}{2} + \tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi,\varpi)}{2}$
$+ \tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi,k(\varpi,\jmath))}{2}$

i.e., $\partial(\varpi, k(\varpi, \jmath)) \preceq_{i_2} \frac{\tau_3}{2}\, \partial(\varpi, k(\varpi, \jmath))$

i.e., $\|\partial(\varpi, k(\varpi, \jmath))\| \le \frac{\tau_3}{2}\, \|\partial(\varpi, k(\varpi, \jmath))\|$

i.e., $(1 - \frac{\tau_3}{2})\, \|\partial(\varpi, k(\varpi, \jmath))\| \le 0$.

Since $1 > \tau_1 + \tau_2 + \tau_3$.

Therefore, we get $\|\partial(\varpi, k(\varpi, \jmath))\| = 0$. Hence $k(\varpi, \jmath) = \varpi$

Similarly, we can easily show that $k(\jmath, \varpi) = \varpi$.

Thus, $(\varpi, \jmath)$ is a common coupled invariant point of h and k.

Now we prove $(\varpi, \jmath)$ is unique.

Let (ℓ, υ) be any other common coupled invariant point of h and k. Then $h(\ell, \upsilon) = k(\ell, \upsilon) = \ell$ and $h(\upsilon, \ell) = k(\upsilon, \ell) = \upsilon$.

Now we consider,

$\partial(\varpi, \ell) = \partial(h(\varpi, \jmath), k(\ell, \upsilon))$
$\preceq_{i_2} \tau_1 \frac{\partial(\varpi,\ell)+\partial(\jmath,\upsilon)}{2} + \tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\ell,\varpi)}{2}$
$+ \tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\ell,k(\ell,\upsilon))}{2}$
$= \tau_1 \frac{\partial(\varpi,\ell)+\partial(\jmath,\upsilon)}{2} + \tau_2 \frac{\partial(\varpi,\varpi)+\partial(\ell,\varpi)}{2} + \tau_3 \frac{\partial(\varpi,\varpi)+\partial(\ell,\ell)}{2}$
$= (\frac{\tau_1+\tau_2}{2})\partial(\varpi, \ell) + \frac{\tau_1}{2}\partial(\jmath, \upsilon)$ —(3.3)

Similarly, we can show that

$\partial(\jmath, \upsilon) \preceq_{i_2} (\frac{\tau_1+\tau_2}{2})\partial(\jmath, \upsilon) + \frac{\tau_1}{2}\partial(\varpi, \ell)$ —(3.4)

By adding the equations (3.3) and (3.4), we get

$\partial(\varpi, \ell) + \partial(\jmath, \upsilon) \preceq_{i_2} (\frac{2\tau_1+\tau_2}{2})[\partial(\varpi, \ell) + \partial(\jmath, \upsilon)]$

i.e., $(1 - \frac{2\tau_1+\tau_2}{2})[\partial(\varpi, \ell) + \partial(\jmath, \upsilon)] \preceq_{i_2} 0$.

Since $1 > \tau_1 + \tau_2 + \tau_3$,

Therefore, we get $\|\partial(\varpi, \ell) + \partial(\jmath, \upsilon)\| \le 0$.

Then we get $\partial(\varpi, \ell) + \partial(\jmath, \upsilon) = 0$.

Hence, $\varpi = \ell$ and $\jmath = \upsilon$. i.e., $(\varpi, \jmath) = (\ell, \upsilon)$.

Hence $(\varpi, \jmath)$ is the one and only one common coupled invariant point of h and k.

Acknowledgements

We are very grateful to experts for their appropriate and constructive suggestions to improve this template.

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Advances and Applications in Mathematical Sciences
Volume 22, Issue 4, February 2023, Pages 883-899

SOME FIXED-POINT RESULTS IN S_b-METRIC SPACES

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Abstract

In this paper, we establish some fixed point and common fixed-point theorems in S_b-metric spaces using implicit relation. The results presented in this paper extend and generalize several results from the existing literature.

1. Introduction

In 1906, Maurice Fréchet [4] introduced the concept of metric spaces. Later, in the year 1922, Stefan Banach [2] proved a very famous theorem called "Banach Fixed Point Theorem". In 2006, Z. Mustafa and B. Sims [5] introduced G-metric spaces. In 2012, Sedghi, Shobe and Aliouche [11] introduced S-metric spaces and they claimed that S-metric spaces are the generalization of G-metric spaces. But, later Dung, Hieu and Radojevic [3] have given examples that S-metric spaces are not the generalization of G-metric spaces or vice versa. Therefore, the collection of G-metric spaces and S-metric spaces are different. In 1989, I. A. Bakhtin [1] introduced b-metric spaces as a generalization of metric spaces. In 2016, N. Souayah, N. Mlaiki [12] introduced S_b-metric spaces as the generalizations of b-metric spaces and S-metric spaces. But, very recently Tas and Ozur [6] studied some relations between S_b-metric spaces and some other metric spaces. S. Sedghi and N. V. Dung [9] introduced an implicit relation to investigate some fixed-

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords: Fixed point, Common fixed point, implicit relation, S_b-metric space.

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Received September 15, 2022; Revised December 18, 2022; Accepted December 19, 2022

point theorems on S-metric spaces. In 2015, Prudhvi [7] proved some fixed-point theorems on S-metric spaces, which extends the results of Sedgi and Dung [9].

Inspired by G. S. Saluja [8], Prudhvi [7], S. Sedghi, N. V. Dung [9] and some others, we establish some fixed point and common fixed-point theorems in S_b-metric spaces satisfying an implicit relation.

2. Preliminaries

Definition 2.1[11]**.** Let Ω be a nonempty set. An S-metric on Ω is a function $S : \Omega^3 \to [0, \infty)$ that satisfies the following conditions, for each $\varsigma, \vartheta, w, a \in \Omega$,

(S1) $S(\varsigma, \vartheta, w) > 0$ for all $\varsigma, \vartheta, w \in \Omega$ with $\varsigma \neq \vartheta \neq w$.

(S2) $S(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w$.

(S3) $S(\varsigma, \vartheta, w) \leq [S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$.

The pair (Ω, S) is called S-metric space.

Example 2.1[3]**.** Let $\Omega = R$, the set of all real numbers and let $S(\varsigma, \vartheta, w) = |\vartheta + w - 2\varsigma| + |\vartheta - w|$ $\forall \varsigma, \vartheta, w \in \Omega$. Then (Ω, S) is an S-metric space.

Definition 2.2[1]**.** Let Ω be a nonempty set. A b-metric on Ω is a function $d : \Omega^2 \to [0, \infty)$ if there exists a real number $s \geq 1$ such that the following conditions holds for all $\varsigma, \vartheta \in \Omega$

(i) $d(\varsigma, \vartheta) = 0 \Leftrightarrow \varsigma = \vartheta$.

(ii) $d(\varsigma, \vartheta) = d(\vartheta, \varsigma)$

(iii) $d(\varsigma, \vartheta) \leq s[d(\varsigma, w) + d(w, \vartheta)]$

The pair (Ω, d) is called a b-metric space.

Definition 2.3[12]**.** Let Ω be a nonempty set and let $s \geq 1$ be a given number.

A function $S_b : \Omega^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $\forall\, \varsigma, \vartheta, w, a \in \Omega,$ the following conditions hold:

(i) $S_b(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w.$

(ii) $S_b(\varsigma, \vartheta, w) \le s[S_b(\varsigma, \varsigma, a) + S_b(\vartheta, \vartheta, a) + S_b(w, w, a)],$

The pair (Ω, S_b) is called an S_b-metric space.

Remark 2.1. We note that every S-metric space is an S_b-metric space with $s = 1,$ but the converse statement is not true.

Example 2.2[6]**.** Let $\Omega = R,$ the set of all real numbers and let $S_b(\varsigma, \vartheta, w) = \frac{1}{16}(|\varsigma - \vartheta| + |\vartheta - w| + |\varsigma - w|)^2,$ for all $\varsigma, \vartheta, w \in \Omega.$

Then (Ω, S_b) is an S_b-metric space with $s = 4,$ but it is not an S-metric space. Indeed, for $\varsigma = 4,$ $\vartheta = 6,$ $w = 8$ and $a = 5,$ we get

$$S_b(4, 6, 8) = 4 > S_b(4, 4, 5) + S_b(6, 6, 5) + S_b(8, 8, 5).$$

Thus, S_b-metric spaces are more general than S-metric spaces.

Definition 2.4[6]**.** A S_b-metric S_b is said to be symmetric if

$$S_b(\varsigma, \varsigma, \vartheta) = S_b(\vartheta, \vartheta, \varsigma)\ \forall\, \varsigma, \vartheta \in \Omega.$$

Lemma 2.1[10]**.** *In S_b-metric space, we have*

(i) $S_b(\varsigma, \varsigma, \vartheta) \le sS_b(\vartheta, \vartheta, \varsigma)$ *and* $S_b(\vartheta, \vartheta, \varsigma) \le sS_b(\varsigma, \varsigma, \vartheta)$

(ii) $S_b(\varsigma, \varsigma, w) \le 2sS_b(\varsigma, \varsigma, \vartheta) + s^2 S_b(\vartheta, \vartheta, w).$

Definition 2.5[12]**.** If (Ω, S_b) is an S_b-metric space and a sequence $\{\varsigma_n\}$ in Ω. Then

(i) $\{\varsigma_n\}$ is called a S_b-Cauchy sequence, if to each $\epsilon > 0, \exists\, n_0 \in N$ such that $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \le \epsilon, \forall\, n, m > n_0.$

(ii) $\{\varsigma_n\} \to \varsigma \Leftrightarrow$ to each $\epsilon > 0, \exists\, n_0 \in N$ such that $S_b(\varsigma_n, \varsigma_n, \varsigma) < \epsilon$ and $S_b(\varsigma, \varsigma, \varsigma_n) < \epsilon\ \forall\, n \ge n_0,$ and we write as $\lim_{n \to \infty} \varsigma_n = \varsigma.$

Definition 2.6[12]**.** We say that (Ω, S_b) is complete if every S_b-Cauchy sequence is S_b-Convergent in Ω.

Tas and Ozgur [6] proved the following theorems in S_b-metric spaces.

Theorem 2.1[6]**.** *If (Ω, S_b) is a complete S_b-metric space with $s \geq 1$ and T is a self map on Ω satisfying*

$$S_b(T\varsigma, T\varsigma, T\vartheta) \leq cS_b(\varsigma, \varsigma, \vartheta), \forall \varsigma, \vartheta \in \Omega, \textit{ where } 0 < c < \frac{1}{s^2}.$$

Then T has a unique fixed point ς in Ω.

Example 2.3[10]**.** Let (Ω, S) be a S-metric space and $S_*(\varsigma, \vartheta, w) = [S(\varsigma, \vartheta, w)]^q$, where $q > 1$ is a real number.

Note that S_* is a S_b-metric with $s = 2^{2(q-1)}$. Obvisously, S_* satisfies conditions

(i) $0 < S_*(\varsigma, \vartheta, w)$, for all $\varsigma, \vartheta, w \in \Omega$ with $\varsigma \neq \vartheta \neq w$.

(ii) $S_*(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w$.

If $1 < q < \infty$, then the convexity of the function $f(\varsigma) = \varsigma^q$, $(\varsigma > 0)$ implies that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$.

Thus, for each $\varsigma, \vartheta, w, a \in \Omega$, we obtain,

$$\begin{aligned} S_*(\varsigma, \vartheta, w) &= S(\varsigma, \vartheta, w)^q \\ &\leq ([S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a)] + S(w, w, a))^q \\ &\leq 2^{q-1}([S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a)]^q + S(w, w, a)^q) \\ &\leq 2^{2-1}([2^{q-1}(S(\varsigma, \varsigma, a)^q + S(\vartheta, \vartheta, a)^q)] + 2^{q-1}S(w, w, a)^q) \\ &\leq 2^{2(q-1)}(S(\varsigma, \varsigma, a)^q + S(\vartheta, \vartheta, a)^q + S(w, w, a)^q). \\ &\leq 2^{2(q-1)}(S_*(\varsigma, \varsigma, a) + S_*(\vartheta, \vartheta, a) + S_*(w, w, a)). \end{aligned}$$

So, S_* is a S_b-metric with $s = 2^{2(q-1)}$.

Now, we introduce an implicit relation to prove some fixed point and common fixed-point theorems in S_b-metric spaces.

Definition 2.7 (Implicit Relation). Let Ψ be the family of all real valued continuous functions $\psi : R_+^5 \to R_+$ non-decreasing in the first argument for five variables. For some $q \in \left[0, \frac{1}{s^2}\right]$, where $s \geq 1$, we consider the following conditions.

(R1) For $\varsigma, \vartheta \in R_+$, if $\varsigma \leq \psi(\vartheta, s\varsigma, s\vartheta, s\varsigma, \varsigma + s\vartheta)$ then $\varsigma \leq q\vartheta$.

(R2) For $\varsigma, \vartheta \in R_+$, if $\varsigma \leq \psi(0, 0, \varsigma, 0, 0)$ then $\varsigma = 0$.

(R3) For $\varsigma \in R_+$, if $\varsigma \leq \psi\left(\varsigma, 0, 0, 0, \frac{\varsigma}{2}\right)$ then $\varsigma = 0$.

3. Main Results

In this section, we shall prove some fixed point and common fixed-point theorems satisfying an implicit relation in S_b-metric spaces.

Theorem 3.1. *Let T be a self map on a complete S_b-metric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T\varsigma, T\vartheta, Tw) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T\varsigma), S_b(w, w, Tw), S_b(\varsigma, \varsigma, T\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T\vartheta) + S_b(w, w, T\varsigma)]) \quad (1)$$

for all $\varsigma, \vartheta, w \in \Omega$ and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and (R3), *then T has a unique fixed point in* Ω.

Proof. Let $\varsigma_0 \in \Omega$ be arbitrary and define a sequence $\{\varsigma_n\}$ in Ω such that $\varsigma_{n+1} = T\varsigma_n$, for any $n \in N$. If for some $n \in N, \varsigma_{n+1} = \varsigma_n$. Then, $\varsigma_n = T\varsigma_n$. Hence, T has a fixed point. Now, we may assume that $\varsigma_{n+1} \neq \varsigma_n$, for all $n \in N$. It follows from inequality (1) and Lemma 2.1, we consider

$$S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) = S_b(T\varsigma_n, T\varsigma_n, T\varsigma_{n-1})$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), S_b(\varsigma_n, \varsigma_n, T\varsigma_n), S_b(\varsigma_{n-1}, \varsigma_{n-1}, T\varsigma_{n-1}),$$

$$S_b(\varsigma_n, \varsigma_n, T\varsigma_n), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, T\varsigma_n) + S_b(\varsigma_{n-1}, \varsigma_{n1}, T\varsigma_{n-1})])$$

$$= \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), S_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n),$$

$$S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1})S_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n)])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), \frac{1}{2s}[sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n)$$

$$+2sS_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n) + sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n)])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), \frac{1}{2s}[2sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) + 2s^2S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1})])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), [S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) + sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1})]) \tag{2}$$

Since $\psi \in \Psi$ satisfies the condition (R1), there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) \leq qS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}) \leq q^n S_b(\varsigma_1, \varsigma_1, \varsigma_0) \tag{3}$$

For $n, m \in N$ with $n < m$, using Lemma 2.1 and equation (3), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2aq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1-s^2q}\right) S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\rho \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \rho$. Now we prove that ρ is a fixed point of T. Again by using inequality (1), we obtain

$$S_b(\varsigma_n, \varsigma_n, T\rho) = S_b(T\varsigma_n, T\varsigma_n, T\rho)$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \rho), S_b(\varsigma_n, \varsigma_n, T\varsigma_n), S_b(\rho, \rho, T\rho),$$

$$S_b(\varsigma_n, \varsigma_n, T\varsigma_n), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, T\varsigma_n) + S_b(\rho, \rho, T\varsigma_n)])$$

$$= \psi(S_b(\varsigma_n, \varsigma_n, \rho), S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), S_b(\rho, \rho, T\rho),$$

$$S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + S_b(\rho, \rho, \varsigma_{n+1})])$$

Letting $n \to \infty$, we get

$$S_b(\rho, \rho, T\rho) \leq \psi(S_b(\rho, \rho, \rho), S_b(\rho, \rho, \rho), S_b(\rho, \rho, T\rho),$$

$$S_b(\rho, \rho, \rho), \frac{1}{2s}[S_b(\rho, \rho, \rho) + S_b(\rho, \rho, \rho)])$$

$$\text{that is, } S_b(\rho, \rho, T\rho) \leq \psi(0, 0, S_b(\rho, \rho, T\rho), 0, 0)$$

Since $\psi \in \Psi$ satisfies the condition (R2), then we get

$$S_b(\rho, \rho, T\rho) \leq qS_b(\rho, \rho, T\rho)$$

$$\text{that is, } (1-q)S_b(\rho, \rho, T\rho) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\rho, \rho, T\rho) = 0$. Hence $T\rho = \rho$.

Thus, ρ is a fixed point of T. Now, we show that fixed point of T is unique.

For this, let ρ^* be another fixed point of T. It follows from inequality (1) and Lemma 2.1, we get

$$S_b(\rho, \rho, \rho^*) = S_b(T\rho, T\rho, \rho^*)$$

$$\leq \psi(S_b(\rho, \rho, \rho^*), S_b(\rho, \rho, T\rho), S_b(\rho^*, \rho^*, T\rho^*),$$

$$S_b(\rho, \rho, T\rho), \frac{1}{2s}[S_b(\rho, \rho, T\rho) + S_b(\rho^*, \rho^*, T\rho)])$$

$$= \psi(S_b(\rho, \rho, \rho^*), S_b(\rho, \rho, \rho), S_b(\rho^*, \rho^*, \rho^*),$$

$$S_b(\rho, \rho, \rho), \frac{1}{2s}[S_b(\rho, \rho, \rho) + S_b(\rho^*, \rho^*, \rho)])$$

$$\leq \psi(S_b(\rho, \rho, \rho^*), 0, 0, 0, \frac{1}{2}S_b(\rho, \rho, \rho^*))$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\rho, \rho, \rho^*) \leq qS_b(\rho, \rho, \rho^*)$$

$$\text{that is, } (1 - q)S_b(\rho, \rho, \rho^*) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\rho, \rho, \rho^*) = 0$. Hence $\rho = \rho^*$. Thus the fixed point of T is unique.

Theorem 3.2. *Let T_1 and T_2 be two selfmaps on a complete S_b-metric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T_1\varsigma, T_1\vartheta, T_2w) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T_1\varsigma), S_b(w, w, T_2w),$$

$$S_b(\varsigma, \varsigma, T_1\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1\vartheta) + S_b(w, w, T_1\varsigma)]) \tag{4}$$

for all $\varsigma, \vartheta, w \in \Omega$ and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and (R3), *then T_1 and T_2 have a unique fixed point in Ω.*

Proof. Let $\varsigma_0 \in X$ be arbitrary and a sequence $\{\varsigma_n\}$ in X defined by $\varsigma_{2n+1} = T_1\varsigma_{2n}$ and $\varsigma_{2n+2} = T_2\varsigma_{2n+1}$, for $n = 0, 1, 2, 3, \ldots$.

It follows from inequality (4) and Lemma 2.1, we have

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) = S_b(T_1\varsigma_{2n}, T_1\varsigma_{2n}, T_2\varsigma_{2n-1})$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, T_2\varsigma_{2n-1}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, T_1\varsigma_{2n})])$$

$$= \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n+1})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})$$

$$+2sS_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}) + sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}),$$

$$\frac{1}{2s}[2sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) + 2s^2S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1})]) \tag{5}$$

Since $\psi \in \Psi$ satisfies the condition $(R1)$, there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) \leq qS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}) \leq q^{2n}S_b(\varsigma_1, \varsigma_1, \varsigma_0) \tag{6}$$

For $n, m \in N$ with $n < m$, by using Lemma 2.1 and equation (6), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2sq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1-s^2q}\right) S_b(\varsigma_0, \varsigma_0, \varsigma_1).$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\sigma \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \sigma$. Now we prove that σ is a common fixed point of T_1 and T_2.

For this Consider,

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, T_1\sigma) = S_b(T_1\varsigma_{2n}, T_1\varsigma_{2n}, T_1\sigma)$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \sigma), S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}) + S_b(\sigma, \sigma, T_1\varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \sigma), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\sigma, \sigma, \varsigma_{2n+1})]) \tag{7}$$

Letting $n \to \infty$, we get

$$S_b(\sigma, \sigma, T_1\sigma) \leq \psi(S_b(\sigma, \sigma, \sigma), S_b(\sigma, \sigma, \sigma), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\sigma, \sigma, \sigma), \frac{1}{2s}[S_b(\sigma, \sigma, \sigma) + S_b(\sigma, \sigma, \sigma)])$$

$$\text{that is, } S_b(\sigma, \sigma, T_1\sigma) \leq \psi(0, 0, S_b(\sigma, \sigma, T_1\sigma), 0, 0)$$

Since $\psi \in \Psi$ satisfies the condition (R2), then we get

$$S_b(\sigma, \sigma, T_1\sigma) \leq qS_b(\sigma, \sigma, T_1\sigma)$$

$$\text{that is, } (1-q)S_b(\sigma, \sigma, T_1\sigma) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\sigma, \sigma, T_1\sigma) = 0$. Hence $T_1\sigma = \sigma$.

Similarly, we can show that $T_2\sigma = \sigma$. This shows that σ is a common fixed point of T_1 and T_2. Now we prove that T_1 and T_2 have a unique common fixed point. For this, let σ^* be another common fixed point of T_1 and T_2. It follows from equation (4) and Lemma 2.1, we have

$$S_b(\sigma, \sigma, \sigma^*) = S_b(T_1\sigma, T_1\sigma, T_2\sigma^*)$$

$$\leq \psi(S_b(\sigma, \sigma, \sigma^*), S_b(\sigma, \sigma, T_1\sigma), S_b(\sigma^*, \sigma^*, T_2\sigma^*),$$

$$S_b(\sigma, \sigma, T_1\sigma), \frac{1}{2s}[S_b(\sigma, \sigma, T_1\sigma) + S_b(\sigma^*, \sigma^*, T_2\sigma)])$$

$$= \psi(S_b(\sigma, \sigma, \sigma^*), S_b(\sigma, \sigma, \sigma), S_b(\sigma^*, \sigma^*, \sigma^*)$$

$$S_b(\sigma, \sigma, \sigma), \frac{1}{2s}[S_b(\sigma, \sigma, \sigma) + S_b(\sigma^*, \sigma^*, \sigma)])$$

$$= \psi\left(S_b(\sigma, \sigma, \sigma^*), 0, 0, 0, \frac{1}{2}S_b(\sigma, \sigma, \sigma^*)\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\sigma, \sigma, \sigma^*) \leq qS_b(\sigma, \sigma, \sigma^*)$$

$$\text{that is, } (1 - q)S_b(\sigma, \sigma, \sigma^*)$$

Since $1 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\sigma, \sigma, \sigma^*) = 0$. Hence $\sigma = \sigma^*$. This shows that σ is the unique common fixed point of T_1 and T_2.

Theorem 3.3. *Let T_1 and T_2 be two continuous selfmaps on a complete Sbmetric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T_1^p\varsigma, T_1^p\vartheta, T_2^pw) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T_1^p\varsigma), S_b(w, w, T_2^pw),$$

$$S_b(\varsigma, \varsigma, T_1^p\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1^p\vartheta) + S_b(w, w, T_1^p\varsigma)]) \tag{8}$$

for all $\varsigma, \vartheta, w \in \Omega$, where p and q are integers and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$, then T_1 and T_2 have a unique fixed point in Ω.

Proof. Since T_1^p and T_2^p satisfies the conditions of Theorem 3.2. Let λ be the common fixed point.

Then, we have $T_1^p\lambda = \lambda \Rightarrow T_1(T_1^p\lambda) = T_1\lambda \Rightarrow T_1^p(T_1\lambda) = T_1\lambda.$

If $T_1\lambda = \lambda_0$, then $T_1^p\lambda_0 = \lambda_0$. So, $T_1\lambda$ is a fixed point of T_1^p.

Similarly, $T_2(T_2^q\lambda) = T_2^q(T_2\lambda) = T_2\lambda$. Now, using equation (8) and Lemma 2.1, we obtain

$$S_b(\lambda, \lambda, T_1\lambda) = S_b(T_1^p\lambda, T_1^p\lambda, T_1^p(T_1\lambda))$$

$$\leq \psi(S_b(\lambda, \lambda, T_1\lambda), S_b(\lambda, \lambda, T_1^p\lambda), S_b(T_1\lambda, T_1\lambda, T_1^p(T_1\lambda)),$$

$$S_b(\lambda, \lambda, T_1^p\lambda), \frac{1}{2s}[S_b(\lambda, \lambda, T_1^p\lambda) + S_b(T_1\lambda, T_1\lambda, T_1^p\lambda)])$$

$$\leq \psi(S_b(\lambda, \lambda, T_1\lambda), S_b(\lambda, \lambda, \lambda), S_b(T_1\lambda, T_1\lambda, T_1\lambda),$$

$$S_b(\lambda, \lambda, \lambda), \frac{1}{2s}[S_b(\lambda, \lambda, \lambda) + S_b(T_1\lambda, T_1\lambda, \lambda)])$$

$$\leq \psi\left(S_b(\lambda, \lambda, T_1\lambda), 0, 0, 0, \frac{1}{2}[S_b(\lambda, \lambda, T_1\lambda)]\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\lambda, \lambda, T_1\lambda) \leq kS_b(\lambda, \lambda, T_1\lambda)$$

$$\text{that is, } (1 - k)S_b(\lambda, \lambda, T_1\lambda) \leq 0.$$

Since $0 \leq k \leq \frac{1}{s^2}$ and $s \geq 1$. Therefore we get $S_b(\lambda, \lambda, T_1\lambda) = 0$. Hence $T_1\lambda = \lambda$. Similarly, we can show that $T_2\lambda = \lambda$. This shows that λ is a common fixed point of T_1 and T_2. For uniqueness of λ, Let $\lambda^* \neq \lambda$ be another common fixed point of T_1 and T_2. Then clearly λ^* is also a common fixed point of T_1^p and T_2^q, which implies $\lambda = \lambda^*$. Hence T_1 and T_2 have a unique common fixed point.

Theorem 3.4. *Let* $\{G_\alpha\}$ *be a family of continuous selfmaps on a complete* S_b*-metric space* (Ω, S_b) *with* $s \geq 1$ *and*

$$S_b(G_\alpha \varsigma, G_\alpha \vartheta, G_\beta w) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, G_\alpha \varsigma), S_b(w, w, G_\beta w),$$

$$S_b(\varsigma, \varsigma, G_\alpha \vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, G_\alpha \vartheta) + S_b(w, w, G_\alpha \varsigma)]) \tag{9}$$

for all $\varsigma, \vartheta, w \in \Omega$, *and* $\alpha, \beta \in R^+$ *with* $\alpha \neq \beta$. *Then there exists a unique* $\eta \in \Omega$ *satisfying* $G_\alpha \eta = \eta$, *for all* $\alpha \in \Psi$.

Proof. Let $\varsigma_0 \in \Omega$ be arbitrary and a sequence $\{\varsigma_n\}$ in Ω defined by $\varsigma_{2n+1} = G_\alpha \varsigma_{2n}$ and $\varsigma_{2n+2} = G_\beta \varsigma_{2n+1}$, for $n = 0, 1, 2, 3, \ldots$.

It follows from inequality (9) and Lemma 2.1, we have

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) = S_b(G_\alpha \varsigma_{2n}, G_\alpha \varsigma_{2n}, G_\beta \varsigma_{2n-1})$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha \varsigma_{2n}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, G_\beta \varsigma_{2n-1})$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha \varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha \varsigma_{2n}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, G_\alpha \varsigma_{2n})])$$

$$= \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n+1})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})$$

$$+2sS_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}) + sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[2sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) + 2s^2 S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1})]) \tag{10}$$

Since $\psi \in \Psi$ satisfies the condition (R1), there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) \leq qS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}) \leq q^{2n}S_b(\varsigma_1, \varsigma_1, \varsigma_0) \quad (11)$$

For $n, m \in N$ with $n < m$, by using Lemma 2.1 and equation (11), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2sq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1 - s^2q}\right)S_b(\varsigma_0, \varsigma_0, \varsigma_1).$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\eta \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \eta$. By the continuity of G_α and G_β, it is clear that $G_\alpha\eta = G_\beta\eta = \eta$. Therefore η is a common fixed point of G_α and G_β, for all $\alpha \in \Psi$. In order to prove the uniqueness, let us take another common fixed point η^* of G_α and G_β, where $\eta \neq \eta^*$. Then using equation (9) and Lemma 2.1, we obtain

$$S_b(\eta, \eta, \eta^*) = S_b(G_\alpha\eta, G_\alpha\eta, G_\beta\eta^*)$$

$$\leq \psi(S_b(\eta, \eta, \eta^*), S_b(\eta, \eta, G_\alpha\eta), S_b(\eta^*, \eta^*, G_\beta\eta^*),$$

$$S_b(\eta, \eta, G_\alpha\eta), \frac{1}{2s}[S_b(\eta, \eta, G_\alpha\eta) + S_b(\eta^*, \eta^*, G_\alpha\eta)])$$

$$\leq \psi(S_b(\eta, \eta, \eta^*), S_b(\eta, \eta, \eta), S_b(\eta^*, \eta^*, \eta^*),$$

$$S_b(\eta, \eta, \eta), \frac{1}{2s}[S_b(\eta, \eta, \eta) + S_b(\eta^*, \eta^*, \eta)])$$

$$\leq \psi\left(S_b(\eta, \eta, \eta^*), 0, 0, 0, \frac{1}{2} S_b(\eta, \eta, \eta^*)\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\eta, \eta, \eta^*) \leq q S_b(\eta, \eta, \eta^*)$$

$$\text{that is, } (1-q) S_b(\eta, \eta, \eta^*) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\eta, \eta, \eta^*) = 0$. Hence $\eta = \eta^*$. This shows that η is the unique common fixed point of G_α, for all $\alpha \in \Psi$.

Corollary 3.1. *Let* (Ω, S_b) *be a complete* S_b*-metric space. Suppose that the mapping* $T : \Omega \to \Omega$ *satisfies* $S_b(T\varsigma, T\vartheta, Tw) \leq \gamma S_b(\varsigma, \vartheta, w)$ *for all* $\varsigma, \vartheta, w \in \Omega$, *where* $\gamma \in [0, 1)$ *is a constant. Then T has a unique fixed point in* Ω. *Moreover, T is continuous at the fixed point.*

Proof. We can prove easily by using Theorem 3.1. with $\psi(a, b, c, d, e) = \gamma a$, for some $\gamma \in [0, 1)$ and $a, b, c, d, e \in R^+$.

Corollary 3.2. *Let* (Ω, S_b) *be a complete* S_b*-metric space. Suppose that the mappings* $T_1, T_2 : \Omega \to \Omega$ *satisfies* $S_b(T_1\varsigma, T_1\vartheta, T_2, w) \leq \delta S_b(\varsigma, \vartheta, w)$ *for all* $\varsigma, \vartheta, \omega \in \Omega$, *where* $\delta \in [0, 1)$ *is a constant. Then* T_1 *and* T_2 *have a unique fixed point in* Ω.

Proof. We can prove easiy by using Theorem 3.2. with $\psi(a, b, c, d, e) = \delta a$, for some $\delta \in [0, 1)$ and $a, b, c, d, e \in R^+$.

Example 3.1. Let (Ω, S_b) be a complete S_b-metric space with $s = 4$. Where $\Omega = [0, 1]$ and $S_b(\varsigma, \vartheta, w) = (|\varsigma - w| + |\vartheta - w|)^2$.

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $T(\varsigma) = \frac{\varsigma}{5}$, for all $\varsigma \in [0, 1]$. Then $S_b(T\varsigma, T\vartheta, Tw) = (|T\varsigma - Tw| + |T\vartheta - Tw|)^2$

$$= \left(\left|\frac{\varsigma}{5} - \frac{w}{5}\right| + \left|\frac{\vartheta}{5} - \frac{w}{5}\right|\right)^2$$

$$= \frac{1}{25}(|\varsigma - w| + |\vartheta - w|)^2$$

$$\leq \frac{1}{25} S_b(\varsigma, \vartheta, w)$$

$$= \gamma S_b(\varsigma, \vartheta, w).$$

where $\gamma = \frac{1}{25} < 1$. Thus T satisfies all the conditions of corollary 3.1. and clearly $0 \in \Omega$ is the unique fixed point of T.

4. Conclusion

From this results, we can study the fixed-circle problem [13] using new contrations on different generalized metric spaces.

5. Acknowledgements

We are very grateful to experts for their appropriate and constructive suggestions to improve this paper.

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Ratio Mathematica Volume 47, 2023

Some fixed point results using (ψ, ϕ)-generalized almost weakly contractive maps in S-metric spaces

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Abstract

Fixed point theorems have been proved for various contractive conditions by several authors in the existing literature. In this article, we define an (ψ, ϕ)- generalized almost weakly contractive map in S-metric spaces and prove an existence and uniqueness of fixed point of such maps. And also we deduce some existing results as special cases of our result. Moreover, we give an example in support of the results.

Keywords: Fixed point; generalized almost weakly contractive map; S-metric space;
2020 AMS subject classifications: 47H10,54H25 [1]

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[1]Received on September 7, 2022. Accepted on March 1, 2021. Published on June 30, 2023.
doi: 10.23755/rm.v4li0.855. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Fixed point technique is considered as one of the powerful tools to solve several problems occur in several fields like Computer science, Economics, Mathematics and its allied subjects. In the year 1906, M.Frechet [7] introduced metric spaces. Later, in the year 1922, Stefan Banach [4] proved a very famous theorem called "Banach Fixed Point Theorem". This theorem has been generalized in many directions by generalizing the underlying space or by viewing it as a common fixed point theorem along with other self maps. In the past few years, a number of generalizations of metric spaces like G -metric spaces, partial metric spaces and cone metric spaces were initiated. These generalizations are used to extend the scope of the study of fixed point theory. In 2012, Sedghi, Shobe and Aliouche [13] introduced S-metric spaces and studied some properties of these spaces. We observe that, every G-metric space need not be a S-metric space and vice-versa. For details, see Examples 2.1 and 2.2 in [5]. Generally, in proving fixed point results for a single self map, we utilize completeness and a contractive condition.

Nowadays, the study of fixed point theorems for self maps satisfying different contraction conditions is the center of rigorous research activities. In this direction, Dutta et al. [6] introduced (ψ, ϕ)-weakly contractive maps in 2008 and obtained some fixed point results for such contractions. Later, G.V.R. Babu et al. [1] introduced (ψ, ϕ)-almost weakly contractive maps in G-metric spaces in 2014. Fixed points of contractive maps on S-metric spaces were studied by several authors [2], [3] and [11]. Since then, several contractions have been considered for proving fixed point theorems.

The main purpose of this paper is to define an (ψ, ϕ)- generalized almost weakly contractive map in S-metric spaces and prove an existence and uniqueness of fixed point of such maps. Furthermore we deduce some results as corollaries to our result and provide an example to validate our result.

2 Preliminaries

Definition 2.1. *[8] A function $\psi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if it satisfies*
(i) ψ is continuous and non decreasing and
(ii) $\psi(t) = 0$ if and only if $t = 0$.
We denote the class of all altering distance functions by Ψ.
We denote $\Phi = \{\phi : [0,\infty) \to [0,\infty) :$ (i) ϕ is continuous and (ii) $\phi(t)=0$ if and only if $t=0\}$.

In the following, Dutta and Choudhury [6] established the fixed points of (ψ, ϕ)-

weakly contractive maps in complete metric spaces.

Theorem 2.1. *[6] Let (X,d) be a complete metric space and let h:X→X be a self-maps of X. If there exist $\psi, \phi \in \Psi$ such that*

$$\psi(d(h\xi, h\vartheta)) \leq \psi(d(\xi, \vartheta)) - \phi(d(\xi, \vartheta)) \;\; for\ all \;\; \xi, \vartheta \in X.$$

Then h has a unique fixed point.

Definition 2.2. *[10] Let X be a non-empty set, G:$X^3 \rightarrow [0, \infty)$ be a function satisfying the following properties:*
(i) $G(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$,
(ii) $G(\xi, \xi, \vartheta) > 0$ for all $\xi, \vartheta \in$X with $\xi \neq \vartheta$,
(iii) G$(\xi, \xi, \vartheta) \leq G(\xi, \vartheta, w)$ for all $\xi, \vartheta, w \in$X,
(iv) $G(\xi, \vartheta, w) = G(\xi, w, \vartheta) = G(w, \xi, \vartheta)$ =...(symmetriy in all three variables),
(v) $G(\xi, \vartheta, w) \leq G(\xi, a, a) + G(a, \vartheta, w)$ for all $\xi, \vartheta, w, a \in$X.
Then the function G is called a generalized metric(G-metric) and the pair (X,G) is called a G-metric space.

Definition 2.3. *[14] Let (X,G) be a G-metric space. A self mapping h of X is said to be weakly contractive if for all $\xi, \vartheta, w \in X$*

$$G(h\xi, h\vartheta, hw) \leq G(\xi, \vartheta, w) - \psi(G(\xi, \vartheta, w))$$

where ψ is an altering distance function.

In 2012, Khandaqji, Al-Sharif and Al-Khaleel [9] proved the following for weakly contractive maps in G-metric spaces.

Theorem 2.2. *[9] Let (X,G) be a complete G-metric space and h:X→X be a self map. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned}\psi(G(h\xi, h\vartheta, hw)) \leq \psi(max\{&G(\xi, \vartheta, w), G(\xi, h\xi, h\xi), G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw),\\ &\alpha G(h\xi, h\xi, \vartheta) + (1-\alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi)\\ &+ (1-\beta)G(\vartheta, h\vartheta, h\vartheta)\}) - \phi(max\{G(\xi, \vartheta, w), G(\xi, h\xi, h\xi),\\ &G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw), \alpha G(h\xi, h\xi, \vartheta)\\ &+ (1-\alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi) + (1-\beta)G(\vartheta, h\vartheta, h\vartheta)\}) \quad (1)\end{aligned}$$

for all $\xi, \vartheta, w \in$X,where $\alpha, \beta \in (0, 1)$. Then h has a unique fixed point u(say) and h is G-continuous at u.

Definition 2.4. *[13] Let a nonempty set X, then we say that a function S:$X^3 \rightarrow [0, \infty)$ is S-metric on X if:*
(S1) $S(\xi, \vartheta, w) > 0$ for all $\xi, \vartheta, w \in$X with $\xi \neq \vartheta \neq w$,
(S2)$S(\xi, \vartheta, w)$ = 0 if $\xi = \vartheta = w$,
(S3) $S(\xi, \vartheta, w) \leq [S(\xi, \xi, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$.
for all $\xi, \vartheta, w, a \in$X. Then (X,S) is called an S-metric space.

Example 2.1. *[13] Let (X,d) be a metric space. Define S:$X^3 \to [0, \infty)$ by $S(\xi, \vartheta, w) = d(\xi, \vartheta) + d(\xi, w) + d(\vartheta, w)$ for all $\xi, \vartheta, w \in$X. Then S is an S-metric on X and S is called the S-metric induced by the metric d.*

Example 2.2. *[5] Let X=R, the set of all real numbers and let $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in$X. Then (X,S) is an S-metric space.*

Example 2.3. *[12] Let X=R, the set of all real numbers and let $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for all $\xi, \vartheta, w \in$X. Then (X,S) is an S-metric space.*

Example 2.4. *Let X=[0,1] and we define S:$X^3 \to [0, \infty)$ by*

$$S(\xi, \vartheta, w) = \begin{cases} 0 & if\, \xi = \vartheta = w \\ max\{\xi, \vartheta, w\} & otherwise \end{cases}.$$

Then S is an S-metric on X.

The following lemmas are useful in our main results.

Lemma 2.1. *[13] In an S-metric space, we have $S(\xi, \xi, \vartheta) = S(\vartheta, \vartheta, \xi)$.*

Lemma 2.2. *[5] In an S-metric space, we have*
(i) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(\vartheta, \vartheta, w)$ and
(ii) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(w, w, \vartheta)$.

Definition 2.5. *[13] Let (X,S) be an S-metric space. We define the following:*
(i) a sequence $\{\xi_n\} \in$X converges to a point $\xi \in$X if $S(\xi_n, \xi_n, \xi) \to 0$ as n $\to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all n$\geq n_0$, $S(\xi_n, \xi_n, \xi) < \epsilon$ and we denote it by $\lim_{n\to\infty} \xi_n = \xi$.
(ii) a sequence $\{\xi_n\} \in$X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $S(\xi_n, \xi_n, \xi_m) < \epsilon$ for all n,m$\geq n_0$.
(iii) (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Definition 2.6. *Let (X,S) and (Y,S') be two S-metric spaces. Then a function h:X→Y is S-continuous at a point $\xi \in$X if it is S-sequentially continuous at ξ, that is, whenever $\{\xi_n\}$ is S-convergent to ξ, we have h(ξ_n) is S'-convergent to h(ξ).*

Lemma 2.3. *[13] Let (X,S) be an S-metric space. If the sequences $\{\xi_n\}$ in X converges to ξ, then ξ is unique.*

Lemma 2.4. *[13] Let (X,S) be an S-metric space. If there exist sequences $\{\xi_n\}$ and $\{\vartheta_n\}$ in X such that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \vartheta_n = \vartheta$, then*
$\lim_{n\to\infty} S(\xi_n, \xi_n, \vartheta_n) = S(\xi, \xi, \vartheta)$.

Definition 2.7. *[13] Let (X,S) be an S-metric space. A map h:X→X is said to be an S-contraction if there exists a constant* $0 \leq \lambda < 1$ *such that*

$$S(h(\xi), h(\xi), h(\vartheta)) \leq \lambda S(\xi, \xi, \vartheta) \ \ for \ all \ \ \xi, \vartheta \in X.$$

We now introduce the following definition and support it with a subsequent example.

Definition 2.8. *Let (X,S) be an S-metric space. A map h:X* $\rightarrow$ *X is called* (ψ, ϕ) *-generalized almost weakly contractive if it satisfies the inequality*

$$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w) \qquad (2)$$

for all $\xi, \vartheta, w \in X$*,* $\psi \in \Psi$*,* $\phi \in \Phi$ *and* $L \geq 0$*, where*
$M(\xi, \vartheta, w) = max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}$,
$\theta(\xi, \vartheta, w) = min\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, h\xi), S(\xi, \xi, hw)\}$.

Example 2.5. *Let X = [0,* $\frac{8}{7}$*] and we define h : X* $\rightarrow$ *X by*

$$h\xi = \begin{cases} \frac{\xi}{10} & if\, \xi \in [0,1] \\ \xi - \frac{4}{5} & if\, \xi \in (1, \frac{8}{7}] \end{cases}.$$

We define S: $X^3 \rightarrow [0, \infty)$ *by* $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ *for all* $\xi, \vartheta, w \in X$*.*
Then (X,S) is a complete S-metric space.
We now define functions $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ *by*

$$\psi(t) = t, \text{ for all } t \geq 0 \text{ and } \phi(t) = \begin{cases} \frac{t}{2} & if\, t \in [0,1] \\ \frac{t}{t+1} & if\, t \geq 1. \end{cases}.$$

We now show that h satisfies the inequality (2).
Case(i): Let $\xi, \vartheta, w \in$ *[0,1].*
Without loss of generality, we assume that $\xi > \vartheta > w$*.*
$S(h\xi, h\vartheta, hw) = S(\frac{\xi}{10}, \frac{\vartheta}{10}, \frac{w}{10}) = \frac{1}{10}(|\xi - w| + |\vartheta - w|)$ *and*
$S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$.
sub case (i): If $|\xi - w| + |\vartheta - w| \in$ *[0,1].*
In this case,

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= \frac{1}{10}(|\xi - w| + |\vartheta - w|) \leq \frac{1}{2}(|\xi - w| + |\vartheta - w|) \\ &= \frac{1}{2}S(\xi, \vartheta, w) \leq \frac{1}{2}M(\xi, \vartheta, w) \\ &= M(\xi, \vartheta, w) - \frac{1}{2}M(\xi, \vartheta, w) \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Sub case(ii): If $|\xi-\vartheta|+|\vartheta-w|\geq$*1.*
In this case,

$$\begin{aligned}
S(h\xi,h\vartheta,hw) &= \frac{1}{10}(|\xi-\vartheta|+|\vartheta-w|) \leq |\xi-\vartheta|+|\vartheta-w| - \frac{|\xi-\vartheta|+|\vartheta-w|}{1+|\xi-\vartheta|+|\vartheta-w|} \\
&= S(\xi,\vartheta,w) - \frac{S(\xi,\vartheta,w)}{1+S(\xi,\vartheta,w)} \\
&= \frac{(S(\xi,\vartheta,w))^2}{1+S(\xi,\vartheta,w)} \leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} \\
&= M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}$$

Case(ii): Let $\xi,\vartheta,w\in(1,\frac{8}{7}]$.
Without loss of generality, we assume that $\xi>\vartheta>w$.

$$\begin{aligned}
S(h\xi,h\vartheta,hw) &= S(\xi-\frac{4}{5},\vartheta-\frac{4}{5},w-\frac{4}{5}) = |\xi-w|+|\vartheta-w| \\
&\leq \frac{2}{7} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\xi,\xi,h\xi) - \frac{S(\xi,\xi,h\xi)}{1+S(\xi,\xi,h\xi)} \\
&= \frac{(S(\xi,\xi,h\xi))^2}{1+S(\xi,\xi,h\xi)} \leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} \\
&= M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}$$

Case(iii): Let $\vartheta,w\in$ *[0,1] and* $\xi\in(1,\frac{8}{7}]$.
Without loss of generality, we assume that $\vartheta>$*w.*

$$\begin{aligned}
S(h\xi,h\vartheta,hw) &= S(\xi-\frac{4}{5},\frac{\vartheta}{10},\frac{w}{10}) = |\xi-\frac{4}{5}-\frac{w}{10}|+|\frac{\vartheta}{10}-\frac{w}{10}| \\
&= \xi-\frac{w}{10}-\frac{4}{5}+\frac{\vartheta-w}{10} = \xi+\frac{\vartheta}{10}-\frac{w}{5}-\frac{4}{5} \\
&= \frac{31}{70} \leq \frac{64}{65} = \frac{8}{5}-\frac{8}{13} = S(\xi,\xi,h\xi) - \frac{S(\xi,\xi,h\xi)}{1+S(\xi,\xi,f\xi)} \\
&= \frac{(S(\xi,\xi,h\xi))^2}{1+S(\xi,\xi,h\xi)} \leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} \\
&= M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}$$

Case(iv): Let w∈ [0,1] and $\xi, \vartheta \in (1, \frac{8}{7}]$.
Without loss of generality, we assume that $\xi > \vartheta$.

$$S(h\xi, h\vartheta, hw) = S(\xi - \frac{4}{5}, \vartheta - \frac{4}{5}, \frac{w}{10}) = |\xi - \frac{4}{5} - \frac{w}{10}| + |\vartheta - \frac{4}{5} - \frac{w}{10}|$$

$$\begin{aligned}
&= \xi + \vartheta - \frac{w}{5} - \frac{8}{5} = \frac{12}{35} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
&= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
&= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
\end{aligned}$$

Case (v): Let $\xi, \vartheta \in$ *[0,1] and* $w \in (1, \frac{8}{7}]$.
Without loss of generality, we assume that $\xi > \vartheta$.

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \frac{\vartheta}{10}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\frac{\vartheta}{10} - w + \frac{4}{5}| \\
&= |\frac{4}{5} - (w - \frac{\xi}{10})| + |\frac{4}{5} - (w - \frac{\vartheta}{10})| = w - \frac{\xi}{10} - \frac{4}{5} + w - \frac{\vartheta}{10} - \frac{4}{5} \\
&= 2w - \frac{\xi + \vartheta}{10} - \frac{8}{5} = \frac{41}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
&= S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
&= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
\end{aligned}$$

Case (vi): Let $\xi \in$ *[0,1] and* $w, \vartheta \in (1, \frac{8}{7}]$.
Without loss of generality, we assume that $w > \vartheta$.

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \vartheta - \frac{4}{5}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\vartheta - w| \\
&= w - \frac{\xi}{10} - \frac{4}{5} + w - \vartheta = 2w - \frac{\xi}{10} - \frac{4}{5} - \vartheta
\end{aligned}$$

$$\leq \frac{27}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)}$$
$$= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)}$$
$$= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)}$$
$$= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).$$

From all the above cases, we conclude that h is an (ψ, ϕ)-generalized almost weakly contraction map on X.

Lemma 2.5. *[5] Let (X,S) be an S-metric space and $\{\xi_n\}$ be a sequence in X such that $\lim_{n\to\infty} S_b(\xi_n, \xi_n, \xi_{n+1})$ = 0. If $\{\xi_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of natural numbers with $n_k > m_k > k$ such that $S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \geq \epsilon$, $S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) < \epsilon$ and*
(i)$\lim_{k\to\infty} S_b(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) = \epsilon$. *(ii)* $\lim_{k\to\infty} S_b(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} S_b(\xi_{m_k}, \xi_{m_k}, \xi_{n_k-1}) = \epsilon$. *(iv)* $\lim_{k\to\infty} S_b(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \epsilon$.

3 Main Results

Theorem 3.1. *Let (X,S) be a complete S-metric space and h: X $\to$ X be a (ψ, ϕ)-generalized almost weakly contractive mapping. Then h has a unique fixed point in X.*

Proof. Let $\xi_0 \in$ X be arbitrary. We define a sequence $\{\xi_n\}$ by h$\xi_n = \xi_{n+1}$, for n = 0,1,2,....
If $\xi_n = \xi_{n+1}$, for some n$\in$ N, then ξ_n is a fixed point of h.
Suppose $\xi_n \neq \xi_{n+1}$, for all n$\in$ N.
Consider,

$$\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) = \psi(S(h\xi_n, h\xi_n, h\xi_{n-1}))$$
$$\leq \psi(max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n),$$
$$\frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\})$$
$$- \phi(max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n),$$
$$\frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\})$$
$$+ L.min\{S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), S(\xi_{n-1}, \xi_{n-1}, h\xi_n), S(\xi_n, \xi_n, h\xi_{n-1})\}$$

$= \psi(max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}),$
$\frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\})$
$- \phi(max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}),$
$\frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\})$
$+ L.min\{S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_{n-1}, \xi_{n-1}, \xi_{n+1}), S(\xi_n, \xi_n, \xi_n)\}$
$= \psi(max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\}) - \phi(max\{S(\xi_n, \xi_n, \xi_{n-1}),$
$S(\xi_n, \xi_n, \xi_{n+1})\}) + L.0$

If max$\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\} = S(\xi_n, \xi_n, \xi_{n+1})$, then we get

$$\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) - \phi(S(\xi_{n+1}, \xi_{n+1}, \xi_n))$$

that is, $\phi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq 0$, which implies that $S(\xi_{n+1}, \xi_{n+1}, \xi_n) = 0$. Then we get $\xi_{n+1} = \xi_n$, which is a contradiction to our assumption that $\xi_n \neq \xi_{n+1}$, for each n.
Therefore, max$\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\} = S(\xi_n, \xi_n, \xi_{n-1})$,
then we get

$$\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_n, \xi_n, \xi_{n-1})) - \phi(S(\xi_n, \xi_n, \xi_{n-1})) \quad (3)$$

that is $\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_n, \xi_n, \xi_{n-1}))$
Therefore we get, $S(\xi_{n+1}, \xi_{n+1}, \xi_n) \leq S(\xi_n, \xi_n, \xi_{n-1})$, for all n and the sequence $\{S(\xi_{n+1}, \xi_{n+1}, \xi_n)\}$ is decreasing and bounded. So, there exists r $\geq$ 0 such that

$$\lim_{n\to\infty} S(\xi_{n+1}, \xi_{n+1}, \xi_n) = r.$$

Letting n $\to \infty$ in equation (3), we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless r = 0.
Hence,

$$\lim_{n\to\infty} S(\xi_{n+1}, \xi_{n+1}, \xi_n) = 0. \quad (4)$$

Now we prove that $\{\xi_n\}$ is a Cauchy sequence. If not, then there exists an $\epsilon > 0$ for which we can find subsequences $\{\xi_{m(k)}\}$ and $\{\xi_{n(k)}\}$ of $\{\xi_n\}$ and increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ such that n(k) is the smallest index for which $n(k) > m(k) > k$,

$$S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) \geq \epsilon \quad (5)$$

Then, we have

$$S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1}) < \epsilon \tag{6}$$

Now,

$$\begin{aligned}\epsilon &\le S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) = S(\xi_{n(k)}, \xi_{n(k)}, \xi_{m(k)})\\ &\le 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1}) + S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1})\\ &\le \epsilon + 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1}) \quad (Using\ equation\ 6)\end{aligned}$$

Letting k$\to \infty$, we get

$$\lim_{k\to\infty} S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) = \epsilon. \tag{7}$$

Also,

$$\begin{aligned}S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) &\le 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + S(\xi_{n(k)}, \xi_{n(k)}, \xi_{m(k)-1})\\ &\le 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1})\\ &\quad + S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1})\end{aligned} \tag{8}$$

and

$$\begin{aligned}S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}) &\le 2S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}) + S(\xi_{n(k)-1}, \xi_{n(k)-1}, \xi_{m(k)})\\ &= 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1})\end{aligned} \tag{9}$$

Letting k $\to \infty$ in equation (9) and using equations (4), (6), (7) and (8) we get

$$\lim_{k\to\infty} S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}) = \epsilon \tag{10}$$

Setting $\xi = \xi_{m(k)-1}, y = \xi_{m(k)-1}$ and $z = \xi_{n(k)-1}$ in equation (2), we obtain

$$\begin{aligned}\psi(\epsilon) &\le \psi(S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)})) = \psi(S(h\xi_{m(k)-1}, h\xi_{m(k)-1}, h\xi_{n(k)-1}))\\ &\le \psi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}),\\ &S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})\\ &+ S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})]\})\\ &- \phi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}),\\ &S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})\\ &+ S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})]\})\end{aligned}$$

$$+ L.min\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}),$$
$$S(\xi_{n(k)-1}, \xi_{n(k)-1}, h\xi_{m(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{n(k)-1})\}$$
$$\leq \psi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}),$$
$$S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})$$
$$+ S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})]\})$$
$$- \phi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}),$$
$$S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})$$
$$+ S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})]\})$$
$$+ L.min\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}),$$
$$S(\xi_{n(k)-1}, \xi_{n(k)-1}, \xi_{m(k)}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)})\}$$

Letting k $\to \infty$ and using equation (10) we get

$$\psi(\epsilon) \leq \psi(max\{\epsilon, 0, 0, 0\}) - \phi(max\{\epsilon, 0, 0, 0\}) + L.min\{0, 0, 0, \epsilon\}$$
$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) + L.0$$

This is a contradiction, since $\epsilon > 0$. This shows that $\{\xi_n\}$ is a Cauchy sequence in the complete S-metric space (X,S). There exists $\kappa \in$X such that $\{\xi_n\} \to \kappa$ as n $\to \infty$.
Now we prove that h$\kappa = \kappa$.
Put $\xi = \xi_n, \vartheta = \xi_n$ and $w = \kappa$ in equation (2), then we get

$$\psi(S(\xi_{n+1}, \xi_{n+1}, f\kappa)) = \psi(S(h\xi_n, h\xi_n, h\kappa))$$
$$\leq \psi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\})$$
$$- \phi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\})$$
$$+ L.min\{S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), S(\kappa, \kappa, h\xi_n), S(\xi_n, \xi_n, h\kappa)\}$$
$$= \psi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), \frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1})$$
$$+ S(\xi_n, \xi_n, \xi_{n+1})]\}) - \phi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, \xi_{n+1}),$$
$$S(\xi_n, \xi_n, \xi_{n+1}), \frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\})$$
$$+ L.min\{S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\kappa, \kappa, \xi_{n+1}), S(\xi_n, \xi_n, h\kappa)\}$$

Letting n $\to \infty$, we get
$\psi(S(\kappa, \kappa, h\kappa)) \leq \psi(S(\kappa, \kappa, \kappa)) - \phi(S(\kappa, \kappa, \kappa)$+L.0
$\psi(S(\kappa, \kappa, h\kappa)) \leq$ 0. So, we get S$(\kappa, \kappa, h\kappa)$=0.

Hence h$\kappa = \kappa$. That is κ is a fixed point of h.
To prove the uniqueness of κ, let j be a fixed point of h with $\kappa \neq$j.
Using equation (2),we consider

$$\begin{aligned}\psi(S(\kappa,\kappa,j)) &= \psi(S(h\kappa,h\kappa,hj))\\ &\leq \psi(max\{S(\kappa,\kappa,j),S(\kappa,\kappa,h\kappa),S(\kappa,\kappa,h\kappa),\frac{1}{2}[S(\kappa,\kappa,h\kappa)+S(\kappa,\kappa,h\kappa)]\})\\ &-\phi(max\{S(\kappa,\kappa,j),S(\kappa,\kappa,h\kappa),S(\kappa,\kappa,h\kappa),\frac{1}{2}[S(\kappa,\kappa,h\kappa)+S(\kappa,\kappa,h\kappa)]\})\\ &+L.min\{S(\kappa,\kappa,h\kappa),S(\kappa,\kappa,h\kappa),S(j,j,h\kappa),S(\kappa,\kappa,hj)\}\\ &That\ is,\ \ \psi(S(\kappa,\kappa,j)) \leq \psi(S(\kappa,\kappa,j)) - \phi(S(\kappa,\kappa,j))\end{aligned}$$

is a contradiction, unless S(κ,κ,j) = 0. Hence we get $\kappa = j$.
This shows that the fixed point of h is unique. □
If L=0 in the Theorem 3.1, then we get the following.

Corollary 3.1. *Let (X,S) be a complete S-metric space and h:X→X be a mapping. Suppose there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned}S(h\xi,h\vartheta,hw) \leq \psi(max\{&S(\xi,\vartheta,w),S(\xi,\xi,h\xi),S(\vartheta,\vartheta,h\vartheta),\frac{1}{2}[S(\xi,\xi,h\vartheta)\\ &+S(\vartheta,\vartheta,h\xi)]\})-\phi(max\{S(\xi,\vartheta,w),S(\xi,\xi,h\xi),S(\vartheta,\vartheta,h\vartheta),\\ &\frac{1}{2}[S(\xi,\xi,h\vartheta)+S(\vartheta,\vartheta,h\xi)]\}),\end{aligned}$$

for all $\xi,\vartheta,w \in X$. Then h has a unique fixed point κ in X.

If ψ is the identity map in the above Corollary (3.1), then we get the following.

Corollary 3.2. *Let (X,S) be a complete S-metric space and h:X→X be a mapping. Suppose there exist $\phi \in \Phi$ such that*

$$\begin{aligned}S(h\xi,h\vartheta,hw) \leq\ &max\{S(\xi,\vartheta,w),S(\xi,\xi,h\xi),S(\vartheta,\vartheta,h\vartheta),\frac{1}{2}[S(\xi,\xi,h\vartheta)+S(\vartheta,\vartheta,h\xi)]\}\\ &-\phi(max\{S(\xi,\vartheta,w),S(\xi,\xi,h\xi),S(\vartheta,\vartheta,h\vartheta),\frac{1}{2}[S(\xi,\xi,h\vartheta)+S(\vartheta,\vartheta,h\xi)]\})\end{aligned}$$

for all $\xi,\vartheta,w \in X$. Then h has a unique fixed point κ in X.

The following example is in support of Theorem 3.1.

Example 3.1. *Let X = [0,$\frac{7}{6}$]. We define S:$X^3 \to [0,\infty)$ by $S(\xi,\vartheta,w) = max\{|\xi-w|,|\vartheta-w|\}$, for all $\xi,\vartheta,w \in X$. Then S is an S-metric on X.*
We define h:X → X by

$$h\xi = \begin{cases} \frac{1}{2} & \textit{if } \xi \in [0,1] \\ \frac{4}{3} - \xi & \textit{if } \xi \in (1, \frac{7}{6}] \end{cases}.$$

We define $\psi, \phi : [0, \infty) \to [0, \infty)$ *by*
$\psi(t)$ *= t, for all t≥ 0 and* $\phi(t) = \frac{t}{1+t}$ *for all t≥0.*
We now show that h satisfies inequality (2).
Case(i) Let ξ, ϑ, w *∈[0,1].*
Without loss of generality, we assume that $\xi > \vartheta > w$.
$S(h\xi, h\vartheta, hw) = S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *= 0. Then trivially the inequality (2) holds.*
Case(ii) Let $\xi, \vartheta, w \in$ *(1,* $\frac{7}{6}]$.
Without loss of generality, we assume that $\xi > \vartheta > w$.

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{4}{3} - w) = max\{|\frac{4}{3} - \xi - (\frac{4}{3} - w)|, |\frac{4}{3} - \vartheta - (\frac{4}{3} - w)|\} \\ &= max\{|w - \xi|, |w - \vartheta|\} = \xi - w \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\ &\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\ &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Case(iii) Let ϑ, w *∈[0,1] and* $\xi \in (1, \frac{7}{6}]$.
Without loss of generality, we assume that $\vartheta > w$.

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{1}{2}, \frac{1}{2}) = max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{1}{2} - \frac{1}{2}|\} \\ &= \xi - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\ &\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\ &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Case(iv) Let w *∈[0,1] and* $\xi, \vartheta \in (1, \frac{7}{6}]$.
Without loss of generality, we assume that $\vartheta > \xi$.

$$S(h\xi, h\vartheta, hw) = S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{1}{2}) = max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{4}{3} - \vartheta - \frac{1}{2}|\}$$

$$
\begin{aligned}
&= max\{|\frac{5}{6}-\xi|,|\frac{5}{6}-\vartheta|\} = \xi-\frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3}-\frac{2}{5} \\
&\leq S(\vartheta,,\vartheta,h\vartheta) - \frac{S(\vartheta,,\vartheta,h\vartheta)}{1+S(\vartheta,,\vartheta,h\vartheta)} = \frac{(S(\vartheta,,\vartheta,h\vartheta))^2}{1+S(\vartheta,,\vartheta,h\vartheta)} \\
&\leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}
$$

Case(v) Let $\xi,\vartheta \in$[0,1] and $w \in (1,\frac{7}{6}]$.
Without loss of generality, we assume that $\xi > \vartheta$.

$$
\begin{aligned}
S(h\xi,h\vartheta,hw) &= (\frac{1}{2},\frac{1}{2},\frac{4}{3}-w) = max\{|\frac{1}{2}-(\frac{4}{3}-w)|,|\frac{1}{2}-(\frac{4}{3}-w)|\} \\
&= w-\frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3}-\frac{2}{5} \\
&\leq S(\xi,\xi,h\xi) - \frac{S(\xi,\xi,h\xi)}{1+S(\xi,\xi,h\xi)} = \frac{(S(\xi,\xi,h\xi))^2}{1+S(\xi,\xi,h\xi)} \\
&\leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}
$$

case(vi) Let $\xi \in$[0,1] and $\vartheta,w \in (1,\frac{7}{6}]$.
Without loss of generality, we assume that $w > \vartheta$.

$$
\begin{aligned}
S(h\xi,h\vartheta,hw) &= S(\frac{1}{2},\frac{4}{3}-\vartheta,\frac{4}{3}-w) = max\{|\frac{1}{2}-(\frac{4}{3}-w)|,|\frac{4}{3}-\vartheta-(\frac{4}{3}-w)|\} \\
&= max\{w-\frac{5}{6},|w-\vartheta|\} = w-\frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3}-\frac{2}{5} \\
&= S(\vartheta,\vartheta,h\vartheta) - \frac{S(\vartheta,\vartheta,h\vartheta)}{1+S(\vartheta,\vartheta,h\vartheta)} = \frac{(S(\vartheta,\vartheta,h\vartheta))^2}{1+S(\vartheta,\vartheta,h\vartheta)} \\
&\leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).
\end{aligned}
$$

case(vii) Let $\vartheta \in$[0,1] and $\xi,w \in (1,\frac{7}{6}]$.
Without loss of generality, we assume that $w > \xi$.

$$
S(h\xi,h\vartheta,hw) = S(\frac{4}{3}-\xi,\frac{1}{2},\frac{4}{3}-w) = max\{|\frac{4}{3}-\xi-(\frac{4}{3}-w)|,|\frac{1}{2}-(\frac{4}{3}-w)|\}
$$

$$= max\{|w-\xi|, w-\frac{5}{6}\} = w-\frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3}-\frac{2}{5}$$
$$= S(\vartheta,\vartheta,hh\vartheta) - \frac{S(\vartheta,\vartheta,h\vartheta)}{1+S(\vartheta,\vartheta,h\vartheta)} = \frac{(S(\vartheta,\vartheta,h\vartheta))^2}{1+S(\vartheta,\vartheta,h\vartheta)}$$
$$\leq \frac{(M(\xi,\vartheta,w))^2}{1+M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1+M(\xi,\vartheta,w)}$$
$$= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)).$$

From all the above cases, we conclude that h is an (ψ, ϕ)-generalized almost weakly contraction map on X and $\frac{1}{2}$ is the unique fixed point of h.

4 Conclusion

In this paper, we establish an existence and uniqueness of a fixed point theorem for (ψ, ϕ)-generalized almost weakly contraction maps in S-metric spaces. As S-metric space is a generalization of metric space, our result in this article extends and improves the result of Khandaqji, Al-Sharif and Al-Khaleel [9] and also generalize several well-known comparable results in the literature. Further, the result in this paper can be extended to several spaces like S_b-metric space, partial S_b-metric spaces and other spaces.

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Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces

Thesis submitted to
Osmania University, Hyderabad

for the award of the degree of
Doctor of Philosophy
in
Mathematics

By

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CERTIFICATE

This is to certify that this thesis entitled "**Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces**" is submitted for the degree of **Doctor of Philosophy** in Mathematics at Osmania University, Hyderabad. It is a genuine research work done by **Mr. D. Venkatesh** in the Department of Mathematics, Osmania University, Hyderabad under my research supervision and this work has not been submitted by him in part or full elsewhere for the award of any degree or diploma.

Place: Hyderabad

Date:

(Prof.V. Naga Raju)

Declaration

I hereby declare that the work which is being presented in this thesis entitled "**Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces**" submitted towards partial fulfilment of the requirements for the award of the degree of **Doctor of Philosophy** in **Mathematics** is an authentic record of my own work carried out under the supervision of **Prof.V. Naga Raju**, Department of Mathematics, Osmania University, Hyderabad.

To the best of my knowledge and belief, this work is no resemblance with any other material previously published except where due reference has been cited in text.

Place: Hyderabad **D.Venkatesh**
Date:

This work is dedicated to my

Parents

&

Teachers

Acknowledgements

I take this opportunity to express my profound sense of gratitude to my research supervisor **Prof.V.Naga Raju**, Department of Mathematics, University College of Science, Osmania University for his inspiring, encouraging guidance, unending benevolence and effusing co-operation in successful completion of my research work.

I am very grateful to **Prof. B. Krishna Reddy**, Head of the Department, Department of Mathematics, Osmania University for extending all facilities for the smooth conduct of my research work.

I am thankful to **Prof.V.Dhanalaxmi**, Chairperson, Board of Studies in Mathematics, **Prof.B.Surender Reddy**, Chairman, Board of Studies in Computer Science and **Prof.N.Kishan**, Former Head, Department of Mathematics, Osmania University for their kind cooperation.

I thank all the faculty members and the office staff of the Department of Mathematics for their unflinching support.

I thanks to my mother **Laxmi Narsamma** and my father **Narsaiah** for their blessings. I thank to my wife **Navya** and my son **Dhanunjay** for their support. I am thankful to all my family members for their continuous help in my research work and providing work environment at my home.

I indebt a lot to **Dr.G.Raja Reddy**, Principal, Kakatiya Government College, Hanamkonda, for their constant encouragement and support.

Special thanks are due to my co-scholars Mr.K.Prasad and Mr.B.Madanlal Naik. The thesis would not have come to successful completion, without their help. Also, I am very much thankful to my friends Mr.D.Suresh and many others who cannot be named here individually for their constant support in my research work. My special words of thanks should also go to my colleagues Dr.B.Prabhakar, Dr.D.Venkanna, Dr.M.Venu gopal and M.Radhika for always being so kind, helpful and motivating.

D.Venkatesh

Abstract

This thesis is in the area of fixed point theory mainly focusing on certain fixed, common and coupled fixed theorems for different contractions in various spaces like S-metric spaces, S_b-metric spaces and bicomplex valued metric spaces. The whole thesis is divided into six chapters.

In Chapter - 1, we give introduction of fixed point theory and provide fundamental definitions, examples, some standard lemmas and properties of metric, b-metric, S-metric, S_b-metric spaces and bicomplex valued metric spaces. We also provide several compatible conditions, common limit in the range (CLR) properties and implicit relations.

In Chapter - 2 of the thesis, we define (ψ, ϕ) - almost weakly generalized contractive map in S-metric spaces and prove the existence and uniqueness of fixed point theorem for such maps. We provide an example to validate our result. This result extends and generalizes a result of Khandaqji, Al-Sharif and Al-Khaleel [111] in G-metric spaces.

In Chapter - 3 of the thesis, we prove a fixed point theorem by defining generalized Z_s-contraction in relation to the simulation function in S-metric spaces. In addition to that, we provide an example which supports our result. The result presented in this chapter generalizes the result of Nihal Tas, Nihal Yilmaz Ozgur and N. Mlaiki [83] in S-metric spaces.

In Chapter - 4 of the thesis, we define (ψ, ϕ) - weakly generalized contractive map in S_b-metric spaces and prove the existence and uniqueness of fixed point theorem for such maps. We also give an example to support of our result. The result presented in this chapter extends and improves the result of GVR. Babu and B.K. Leta [87] in S-metric spaces.

In Chapter - 5 of the thesis, we define an implicit relation in S_b-metric spaces and prove some fixed and common fixed-point theorems in S_b-metric spaces. The

results presented in this chapter extend and generalize the results of GS Saluja [110] in S-metric spaces.

In Chapter- 6 of the thesis, we establish three unique common fixed point theorems for two self-mappings, four self-mappings and six self-mappings in the bicomplex valued metric spaces. Firstly, we generate a common fixed point theorem for four self-mappings by using weaker conditions such as weakly compatibility and CLR_{AB} property. Secondly, we generate a common fixed point theorem for six self-mappings by using inclusion relation, generalized contraction, weakly compatible and commuting maps. Finally, we generate a common coupled fixed point for two self mappings using a generalized contraction in the bicomplex valued metric space.

Thus, in this research work, we investigate the existence and uniqueness of fixed, common and coupled fixed points satisfying specific contractive conditions in various spaces.

Contents

List of Notations

etal.	*et alii (and others)*
i.e.	*id est (that is)*
w.l.o.g.	*wihout loss of generality*
iff	*if and only if*
CLR	*Common Limit in the Range*
lim	*limit*
max	*maximum*
min	*minimum*
inf	*infimum*
sup	*supremum*
$\mathbb{N}$	*The set of all Natural numbers*
$\mathbb{R}$	*The set of all real numbers*
$\mathbb{R}_+$	*The set of non negative real numbers*
$\mathbb{R}^+$	*The set of positive real numbers*
$\mathbb{R}^n$	*The set of n tuples of real numbers*
$\mathbb{C}$	*The set of all Complex numbers*
$\mathbb{C}_2$	*The set of all Bicomplex numbers*

Chapter 1

Introduction

1.1 Introduction and Preliminaries

Fixed point theory is a very broad topic of mathematical research and it has extensive applications in various fields of Mathematics such as: Classical Analysis, Functional Analysis, Operator Theory, Topology, Algebraic Topology, Approximation Theory, Successive Approximation, Integral Equations, Differential Equations, Functional Equations, Variational Inequalities and several others. The origin of fixed point theory dates to the later part of the nineteenth-century heavily rests on the use of successive approximations to establish the existence and uniqueness of solutions, particularly to the differential equations. Fixed point results are also used to study the optimal control problems of some nonlinear systems. In fact, fixed point results on ordered metric spaces provide us exact or approximate solutions of boundary value problems. The theory of fixed points also serves as a bridge between Analysis and Topology besides facilitating a very useful area of interaction between the two.

The fixed point theory continues to be a young area of research despite having a history of more than hundred years. The strength of fixed point theory lies in its applications which is scattered throughout the existing literature of fixed point theory. Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. For example, in theoretical economics, such as general equilibrium theory, a situation arises where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under what conditions will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorems. Hence finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect. In the field of metric fixed point theory, the first important and significant result was proved for contraction mapping in complete metric space by Banach [1] in 1922. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics.

As expected, a self-mapping h on a nonempty set X is said to have a fixed point $\xi \in$ X, if ξ remains fixed under h (i.e., $h\xi = \xi$). In order to illustrate this fact, let us consider the simple quadratic equation $\xi^2 - 5\xi + 6 = 0$. Clearly, $\xi = 2$ and $\xi = 3$ are the roots of this equation. Also, we can rewrite this equation in the following form:

$$\xi = \tfrac{\xi^2+6}{5}.$$

If we think of a real valued map, then the above equation reduces to h$\xi = \xi$. We notice that $\xi = 2$ and $\xi = 3$ are the two fixed points of h. Thus, from the above observations, we conclude that the problem of finding a solution of the functional equation h$\xi - \xi = 0$, is the same as finding the fixed point of the mapping h. A self-mapping h on a nonempty set X can have no fixed point, unique fixed point, a finite number of fixed points and infinitely many fixed points as given below:

1.1.1 Example: Let X=$\mathbb{R}$ (The set of all Real numbers) and h:X$\to$X be a mapping defined as:

(i) h$\xi = \xi$+a, for a$\neq$0;

(ii) h$\xi = \frac{\xi}{2}$;

(iii) h$\xi = \xi^2$;

(iv) h$\xi = \xi$.

Notice that, in example (i) h has no fixed point, (ii) h has unique fixed point (namely 0), (iii) h has two (finite) fixed points (namely 0 and 1) and (iv) h has infinite fixed points (namely $\mathbb{R}$).

Historically, the origin of fixed point theory was effectively utilized to establish the existence and uniqueness of a solutions of differential equations at the end of the nineteenth century. This method can be traced back to the mathematical activities of great mathematicians, such as Cauchy [2], Liouville [3], Lipschitz [4], Peano [5], Picard [6] and some others. But formally, it was started as an important part of analysis in the pioneering work of the great French mathematician Poincaré [7]. By now, there exists an extensive literature on this topic and continues to be a very active domain of research. The investigation of fixed points for several classes of mappings is still on. Though the existence or non existence of a fixed point is an intrinsic property of a mapping, there do exist many necessary or sufficient conditions for the existence of fixed points involving a mixture of topological, order-theoretic or geometric properties on the mapping or in its domain. Fixed point theory is broadly divided into the following three major areas:

1. Topological fixed point theory.
2. Discrete fixed point theory.
3. Metric fixed point theory.

It is not possible to give a comprehensive illustration of core results of a wide and extensive subject like fixed point theory in few paragraphs. However, for a comprehensive study of fixed point theory and its related results, one can consult classical books of Goebel and Kirk [8], Khamsi and Kirk [9], Kirk and Sims [10], Singh et al. [11], Agarwal et al. [12], Dugundji and Granas [13] and Smart [14].

The results of the present thesis fall in the domain of metric fixed point theory. So, in the next section we will talk about this branch of fixed point theory.

The origin of metric fixed point theory is often traced back to the classical Banach contraction principle which was originated in the Ph.D. thesis of the great Polish mathematician Banach [1], in 1922. This principle remains the most versatile elementary result in metric fixed point theory. Metric fixed point theory is comprised of such fixed point results in which the ambient space is equipped with some distance function and the geometric properties of the underlying mappings are effectively utilized. This theory is relatively not new in the functional analysis but still a very active area of research. Before presenting the Banach contraction principle, we recall some relevant notions utilizing geometrical properties of underlying mappings defined on a metric space.

1.1.2 Definition: Let (X, d) be a metric space. A mapping h : X→ X is called
(i) isometry if $d(h\xi, h\vartheta) = d(\xi, \vartheta)$, for all $\xi, \vartheta \in$ X,
(ii) Lipschitzian (or α-Lipschitzian) if there exists $\alpha > 0$ such that

$$d(h\xi, h\vartheta) \leq \alpha\ d(\xi, \vartheta) \quad \forall \xi, \vartheta \in X,$$

(iii) non-expansive if h is 1-Lipschitzian
(iv) expansive if there exists $\alpha > 1$ such that $d(h\xi, h\vartheta) \geq \alpha\ d(\xi, \vartheta)$ $\forall\ \xi, \vartheta \in$ X,

(v) contractive if $d(h\xi, h\vartheta) < d(\xi, \vartheta)$ $\forall \xi, \vartheta \in$X, $\xi \neq \vartheta$,

(vi) contraction (or linear contraction or α-contraction) if h is α-Lipschitzian with $\alpha \in$[0, 1) such that

$$d(h\xi, h\vartheta) \leq \alpha d(\xi, \vartheta) \quad \forall \xi, \vartheta \in X.$$

1.1.3 Definition: Let X be a nonempty set, h be a self-mapping on X and $\xi_0 \in$ X. A sequence $\{\xi_n\} \subset$ X is called Picard's sequence of h based at ξ_0 if $\xi_n = h\xi_{n-1} = h^n\xi_0$ $\forall n \in \mathbb{N}$.

The following result is known in the literature as Banach contraction principle and remains the most versatile elementary result in metric-theoretical fixed point theory.

1.1.4 Theorem: [1] Every contraction mapping on a complete metric space has a unique fixed point.

Moreover, the classical Banach contraction principle guarantees that Picard's sequence of h based at any point converges to the fixed point, i.e., starting at any point $\xi_0 \in$ X, the repeated iterations of the mapping at ξ_0 yields a sequence that converges to the unique fixed point of h. The advantage of this principle is that its hypotheses is very simple and always gives a unique fixed point which can be found using a straightforward method. The only disadvantage attached to this principle is that assuming the mapping to be contraction forces the mapping h to be continuous at each point of the space. However, this principle is widely considered as the source of metric fixed point theory and one of the most fundamental and powerful tools of nonlinear analysis.

Coincidence and Common fixed Point Theory

Let X be a non empty set. Recall that an element $\xi \in$ X is said to be a fixed point of a self-mapping h on X if h$\xi = \xi$. This equation can be written as $h\xi = I\xi$ (where I denotes the identity mapping on X). This observation raised an obvious question: under what conditions can one replace the identity mapping by another self-mapping g on X such that $h\xi = g\xi$? The answer to this question opened a new door towards a new type of activity in fixed point theory under the umbrella of theory of coincidence points. Henceforth, given a pair of self-mappings (h, g) defined on X, consider the following problem regarding to find ξ, $\xi^* \in$ X such that

$$h\xi = g\xi = \xi^*.$$

Then

- ξ is called a coincidence point of the pair (h, g),
- ξ^* is called a point of coincidence of the pair (h, g),
- ξ is called a common fixed point of the pair (h, g), if $\xi = \xi^*$.

Notice that, every common fixed point of the pair (h, g) is also a coincidence point as well as point of coincidence.

In 1967, Machuca [15] proved the earliest metrical coincidence theorem for a pair of mappings h, g : X $\to$ Y, where X and Y are complete metric spaces and T_1-topological space satisfying the first axiom of countability respectively. By taking Y = X, in Machuca coincidence theorem we get the following theorem.

1.1.5 Theorem: Let (X, d) be a complete metric space and h and g be two self-mappings on X. Suppose that the following conditions hold:
(i) h(X) $\subset$ g(X),
(ii) there exists $\alpha \in [0, 1)$ such that

$$d(h\xi, h\vartheta) \leq \alpha \; d(g\xi, g\vartheta) \quad \forall \xi, \vartheta \in X,$$

(iii) either h(X) or g(X) is closed.
Then f and g have a coincidence point.

The condition (iii) was only used to guarantee that (h(X), d) or (g(X), d) is a complete metric space. In 1968, Goebel [16] observed that the condition "h(X) (or g(X)) is complete without assuming the completeness of X" is relatively weaker than "(X, d) is complete and h(X) (or g(X)) is closed" and utilized the same to extend Machuca coincidence theorem for two mappings h, g : X $\to$ Y, where X and Y are complete metric space and an arbitrary set respectively. On taking Y = X, Goebel coincidence theorem runs as:

1.1.6 Theorem: Let (X, d) be a complete metric space and h and g two self-mappings on X. Suppose that the following conditions hold:
(i) h(X) $\subset$ g(X),
(ii) there exists $\alpha \in$ [0, 1) such that

$$d(h\xi, h\vartheta) \leq \alpha\ d(g\xi, g\vartheta) \quad \forall \xi, \vartheta \in X,$$

(iii) either h(X) or g(X) is a complete subspace of X.
Then h and g have a coincidence point.

Fixed point theorems are statements containing sufficient conditions that ensure the existence of a fixed point. Therefore, one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which guarantee a fixed point or a common fixed point as the case may be. Common fixed point theorems for contractive type mappings necessarily require a commutativity condition, a condition on the ranges of the mappings, continuity of one or more mappings besides a contractive condition. And every significant fixed point or common fixed point theorem attempts to weaken or obtain a necessary version of one or more of the these conditions [26].

In 1976, using condition (i) of Theorem 1.1.6., Jungck [25] obtained common fixed point for commuting mappings by using a constructive procedure of sequence of iterates.

1.1.7 Theorem: [25] Let (X, d) be a complete metric space and let h and g be commuting self-maps of X satisfying the conditions:
(i) hX $\subseteq$ gX;
(ii) $d(h\xi, h\vartheta) \leq \alpha d(g\xi, g\vartheta)$, for all $\xi, \vartheta \in$X and some $0 \leq \alpha < 1$.
If g is continuous then h and g have a unique common fixed point.

The essence of Jungck's theorem has been used by several researchers to obtain interesting common fixed point theorems for both commuting and non-commuting pairs of mappings satisfying contractive type conditions. The constructive technique of Jungck's theorem has been further improved and extended by various researchers to establish common fixed point theorems for three mappings, four mappings and sequence of mappings (see also [[27]-[33]]).

Generalizations of Jungck's contraction condition have been extensively used to study common fixed points of contractive mappings. If h and g are two self-mappings of a metric space (X, d), general contractive conditions assume the following form.

(a) ϕ-type contractive condition (in the sense of Boyd and Wong [34]);

$$d(h\xi, h\vartheta) \leq \phi \; d(g\xi, g\vartheta),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that ϕ is upper semi-continuous from the right and $\phi(t) < t$ for each $t > 0$.

(b) Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq d(g\xi, g\vartheta) < \epsilon + \delta \implies d(h\xi, h\vartheta) < \epsilon.$$

Condition (b) is also referred to as a Meir-Keeler type (ϵ, δ) contractive condition [35]. It can easily be seen that if h and g satisfy (b) then h and g also satisfy the contractive condition

$$d(h\xi, h\vartheta) < d(g\xi, g\vartheta).$$

In some results the contractive condition (b) has been replaced by a slightly weaker contractive condition of the following form.

(c) Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon < d(g\xi, g\vartheta) < \epsilon + \delta \implies d(h\xi, h\vartheta) \leq \epsilon.$$

Jachymski [36] has shown that the contractive condition (c) implies (b) but not conversely.

In the setting of common fixed point theorems, the Meir-Keeler type (ϵ, δ) contractive condition alone is not sufficient to guarantee the existence of a common fixed point. While assuming the (ϵ, δ) contractive condition, the existence of a common fixed point is ensured either by imposing some additional restriction on δ or by assuming some additional condition besides the (ϵ, δ) contractive condition or by imposing strong conditions on the continuity of mappings (for references see [[37] - [44]]).

In 1982, Sessa gave the weaker version of the commutativity condition, namely the weakly commuting condition. In subsequent years Jungck [[18],[19]], Tivari and Singh [45], Pathak [[46], [47]], Jungck et al. [48], Jungck and Pathak [49], Pant [50], Pathak et al. [51], Al-Thagafi and Shahzad [52], Hussain et al. [53], Pant and Bisht [54], Bisht and Shahzad [55] and many others have considered several generalizations of commuting mappings or weaker notions of commutativity.

The first ever attempt to relax the commutativity of mappings to a smaller subset of the domain of mappings was initiated by Sessa [17] who in 1982 gave the notion of weak commutativity.

1.1.8 Definition: (Sessa [17]) Two self-mappings h and g of a metric space (X, d) are called weakly commuting iff $d(hg\xi, gh\xi) \leq d(h\xi, g\xi)$ for all ξ in X.

Notice that commuting mappings are obviously weakly commuting. However, a weakly commuting mappings need not be commuting.

1.1.9 Example: Let X = [0, 1] be equipped with the usual metric d on X. Define constant mappings h and g : X→X by

$$h\xi = a \;\; and \;\; g\xi = b, \;\; a \neq b.$$

Then h and g are weakly commuting but not commuting
since $d(hg\xi, gh\xi) = |a - b| = d(h\xi, g\xi)$.

In 1986, Jungck generalized the concept of weak commutativity by introducing the notion of compatible mappings [18] also called asymptotically commuting mappings by Tivari and Singh [45] in an independent work. In [32] it has been shown that two continuous self-mappings of a compact metric space are compatible iff they commute on their set of coincidence points.

1.1.10 Definition: (Jungck [18], Tivari and Singh [45]) Two self-mappings h and g of a metric space (X, d) are called compatible or asymptotically commuting if and only if $lim_{n\to\infty} d(hg\xi_n, gh\xi_n) = 0$, whenever $\{\xi_n\}$ is a sequence in X such that $lim_{n\to\infty} h\xi_n = lim_{n\to\infty} g\xi_n = t$ for some t in X.

Clearly, weakly commuting mappings are compatible, but the converse does not hold.

1.1.11 Example: [18] Let X $=$ $[0,\infty)$ and d be the usual metric on X. Define h , g : X→X by $h\xi = \xi^3$ for all ξ and $g\xi = 2\xi^3$ for all ξ.

Then $d(hg\xi, gh\xi) > d(h\xi, g\xi)$. Therefore h and g are not weakly commuting mappings. However, h and g are compatible mappings.

1.1.12 Remark: Notice that the notions of weak commutativity and compatibility differ in one respect. Weak commutativity is essentially a point property, while the notion of compatibility uses the machinery of sequences.

Ever since the introduction of compatibility, the study of common fixed points has developed around compatible maps and its weaker forms [[66],[67]] and it has become an area of vigorous research activity. However, fixed point theory for non compatible mappings is equally interesting and Pant [56] has initiated some work along these lines. One can establish fixed point theorems for such mappings pairs not only under non expansive conditions but also under Lipschitz type conditions even without using the usual contractive method of proof. The best examples of non compatible maps are found among pairs of mappings which are discontinuous at their common fixed point [56]. It may be observed that the mappings h and g are said to be non compatible if there exists a sequence $\{\xi_n\}$ in X such that for some t in X but $lim_{n\to\infty} d(hg\xi_n, gh\xi_n)$ is either non-zero or nonexistent.

1.1.13 Definition: [57] Two self-mappings h and g of a metric space (X, d) are said to satisfy the (E.A.) property if there exists a sequence $\{\xi_n\}$ in X such that $lim_{n\to\infty} h\xi_n = lim_{n\to\infty} g\xi_n = t$ for some t $\in$ X.

If h and g are both non compatible then they do satisfy the (E.A.) property. In fact the notion of the (E.A.) property circumvents the most crucial part of fixed point theorems consisting of constructive procedures yielding a Cauchy sequence. On the other hand the (E.A.) property enables us to study the existence of common fixed point of non expansive or Lipschitz type conditions in the setting of non complete metric spaces.

Sintunavarat and Kumam [58] introduced an interesting property, namely the common limit in the range property (in short CLRg) which completely buys the condition of closedness of the ranges of the involved mappings and has an edge over the (E.A.) property (see also [[59]-[64]]).

1.1.14 Definition: [[58],[65]] Two self-mappings h and g of a metric space (X, d) are said to be satisfy the common limit in the range of g property (CLRg) if there exists a sequence ξ_n in X such that $lim_{n\to\infty} h\xi_n = lim_{n\to\infty} g\xi_n = g\xi$ for some $\xi \in$ X.

It is important to note that in the setting of metric spaces, there is no general method for the study of common fixed points of non expansive or Lipschitz type mappings. The notions of non compatibility, the (E.A.) property and CLRg property are well suited for studying common fixed points of strict contractive conditions, non expansive type mapping pairs or Lipschitz type mapping pairs in ordinary metric spaces, which are not even complete.

1.1.15 Definition: A pair of self-mappings (h, g) defined on a nonempty set X is said to be commuting if $h(g\xi) = g(h\xi)$ for all $\xi \in$ X.

1.1.16 Definition: [19] Let (h, g) be a pair of a self-mappings on a metric space (X, d). Then the pair (h, g) is said to be weakly compatible if
$h\xi = g\xi \implies g(h\xi) = h(g\xi)$.

Now, we present the following lemma which is used in the sequel.

1.1.17 Lemma: [20] Let (h, g) be a pair of weakly compatible self-mappings defined on a nonempty set X. Then every point of coincidence of the pair (h, g) remains a coincidence point.

Metric Spaces

In 1906, Maurice Rene Frechet introduced the basic notion of metric spaces in his doctoral dissertation [21] submitted to Paris University.

1.1.18 Definition: [21] Let X be a nonempty set. The mapping d:X $\times$ X $\to [0, \infty)$ is said to be a metric on X, if it satisfies the following
($\forall \xi, \vartheta, w \in$ X):
(i) $d(\xi, \vartheta) \geq 0 \ \forall \xi, \vartheta \in X$
(ii) $d(\xi, \vartheta) = 0$ if and only if $\xi = \vartheta$; (identity of indiscernibles)
(iii) $d(\xi, \vartheta) = d(\vartheta, \xi)$; (symmetry)
(iv) $d(\xi, \vartheta) \leq d(\xi, w) + d(w, \vartheta)$. (triangle inequality)
The set X together with a metric d is called metric space and is often denoted by (X, d). If there is no confusion likely to occur, we sometimes, denote the metric spaces (X, d) by X.

1.1.19 Example: Let X = $\mathbb{R}$, the set of all real numbers. Define $d : X \times X \to [0, \infty)$ by $d(\xi, \vartheta) = |\xi - \vartheta|, \forall \xi, \vartheta \in X$.
Then the pair (X, d) is a metric space and the metric d is called the usual metric on $\mathbb{R}$.

1.1.20 Definition: [21] A sequence $\{\xi_n\}$ in (X, d) is said to be convergent to $\xi \in$ X if and only if $lim_{n \to \infty} d(\xi_n, \xi) = 0$.

1.1.21 Definition: [21] A sequence $\{\xi_n\}$ in (X, d) is said to be Cauchy if and only if $lim_{n,m \to \infty} d(\xi_n, \xi_m) = 0$.

1.1.22 Definition: [21] A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

b-Metric Spaces

In 1989, I.A. Bakhtin [22] and S. Czerwik [23] introduced the concept of b-metric space as a noted improvement of metric spaces.

1.1.23 Definition: [[22], [23]] Let X be a nonempty set with s $\geq$ 1. The mapping σ: X $\times$ X $\to$ [0, 1) is said to be a b-metric on X, if it satisfies the following ($\forall \xi, \vartheta, w \in$ X):
(i) $\sigma(\xi, \vartheta) = 0$ if and only if $\xi = \vartheta$;
(ii) $\sigma(\xi, \vartheta) = \sigma(\vartheta, \xi)$;
(iii) $\sigma(\xi, \vartheta) \leq s[\sigma(\xi, w) + \sigma(w, \vartheta)]$.
Then the pair (X, σ) is said to be a b-metric space.

1.1.24 Example: Let X = $\mathbb{R}$, the set of all real numbers. For any $\xi, \vartheta \in$ X, define $\sigma(\xi, \vartheta) = |\xi - \vartheta|^2$. Then the pair (X, σ) is a b-metric space with s = 2.

1.1.25 Definition: [23] A sequence $\{\xi_n\}$ in (X, σ) is said to be convergent to $\xi \in X$ if and only if $lim_{n\to\infty} d(\xi_n, \xi) = 0$.

1.1.26 Definition: [23] A sequence $\{\xi_n\}$ in (X, σ) is said to be Cauchy if and only if $lim_{n,m\to\infty} \sigma(\xi_n, \xi_m) = 0$.

1.1.27 Definition: [23] A b-metric space (X, σ) is said to be complete if every Cauchy sequence in X converges to a point in X.

The following example shows that a general a b-metric is not a continuous mapping.

1.1.28 Example: [24] Let X = N $\cup$ ∞ and a mapping $\sigma : X \times X \to [0, \infty)$ defined by:

$$\sigma(\xi, \vartheta) = \begin{cases} 0 & \text{if } \xi = \vartheta \\ |\frac{1}{\xi} - \frac{1}{\vartheta}| & \text{if } \xi, \vartheta \; are \; even \; or \; \xi\vartheta = \infty \\ 5 & \text{if } \xi, \vartheta \; are \; odd \; or \; \xi \neq \vartheta \\ 5 & \text{if } otherwise. \end{cases}$$

Then the pair (X, σ) is a b-metric space with s=3 but it is not continuous.

In an attempt to generalize fixed point theorems proved for self maps of metric spaces, B.C.Dhage [68] has introduced generalized metric spaces called D-metric spaces as follows:

1.1.29 Definition: A nonempty set X, together with a function $D : X^3 \to [0, \infty)$ is called a D-metric space, denoted by (X, D) if D satisfies

(i) $D(\xi, \vartheta, w) = 0$ if and only if $\xi = \vartheta = w$ (coincidence),
(ii) $D(\xi, \vartheta, w) = D(p\{\xi, \vartheta, w\})$, where p is a permutation of ξ, ϑ, w (symmetry),
(iii) $D(\xi, \vartheta, w) \leq D(\xi, \vartheta, a) + D(\xi, a, w) + D(a, \vartheta, w)$ for all $\xi, \vartheta, w, a \in X$ (tetrahedral inequality).
The non negative real function D is called a D-metric on X.

Subsequently several researchers made significant contribution to the fixed point theory for self maps of D-metric spaces in [69],[70],[71],[72] and [73]. Unfortunately most of the claims concerning the topological structures of D-metric spaces were proved to be by incorrect by S.V.R.Naidu and others in [74], [75] and [76].

As a probable modification to D-metric spaces, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [77] introduced D*-metric spaces. In 2006, Zead Mustafa and Brailey Sims [78] initiated G - metric spaces. While, Shaban Sedghi, Nabi Shobe and Abdelkrim Aliouche [79] introduced S - metric spaces. It was claimed in [79] that (i) every G - metric space is a D*-metric space (ii) every D*-metric space is an S-metric space and therefore (iii) every G - metric space is an S -metric space. We observe by means of an examples in the next two sections that although (ii) is correct, (i) and (iii) are not.

The generalized metric spaces

In this section we give the definitions of the three generalized metric spaces and provide some examples in each case.

1.1.30 Definition: [77] Let X be a non-empty set. A function $D^* : X^3 \to [0, \infty)$ is said to be a D*-metric on X, if it satisfies the conditions:
(i) $D^*(\xi, \vartheta, w) = 0$ if and only if $\xi = \vartheta = w$.
(ii) $D^*(\xi, \vartheta, w) = D^*(\sigma(\xi, \vartheta, w))$ for all $\xi, \vartheta, w \in$ X
where $\sigma(\xi, \vartheta, w)$ is a permutation of the set $\{\xi, \vartheta, w\}$
(iii) $D^*(\xi, \vartheta, w) \leq D^*(\xi, \vartheta, z) + D^*(z, w, w)$ for all $\xi, \vartheta, w, z \in$ X .
A set X with D*-metric is called a D*-metric space and it is denoted by (X, D*).

1.1.31 Example: Let X $=\{a, b, c\}$ and $D^* : X^3 \to [0, \infty)$ be defined by

$$D^*(\xi,\vartheta,w) = \begin{cases} 0 & \text{if } \xi = \vartheta = w \\ \frac{1}{2} & \text{if } \xi,\vartheta,w \text{ are distinct} \\ 1 & \text{if } otherwise. \end{cases}.$$

For instance

(1.1.32.) $D^*(a, b, c) = \frac{1}{2}$ while $D^*(a, a, b) = 1$

It is easy to see that (X, D*) is a D*- metric space, in fact conditions (i) and (ii) of Definition 1.1.30 are trivial. Also if w = z condition (iii) of Definition 1.1.30. holds obviously; In case w $\neq$ z, we have sub cases of x = y = z; and $\xi \neq \vartheta$ with $z \in \{\xi, \vartheta\}$ and $z \notin \{\xi, \vartheta\}$ and in each case (iii) can be verified easily.

1.1.33 Remark: It was shown in ([78], Remark 1.2) that $D^*(\xi, \xi, \vartheta) = D^*(\xi, \vartheta, \vartheta)$ for all $\xi, \vartheta \in X$.

Zead Mustafa and Brailey sims introduced the generalized metric(G-metric) spaces in 2006.

1.1.34 Definition: [78] Let X be a non-empty set and $G : X^3 \to [0, \infty)$ be a function satisfying:

(G1) $G(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$

(G2) $0 < G(\xi, \xi, \vartheta)$ for all $\xi, \vartheta \in X$ with $\xi \neq \vartheta$

(G3) $G(\xi, \xi, \vartheta) \leq G(\xi, \vartheta, w)$ for all $\xi, \vartheta, w \in X$ with $\vartheta \neq w$

(G4) $G(\xi, \vartheta, w) = G(\sigma(\xi, \vartheta, w))$ for all $\xi, \vartheta, w \in X$, where $\sigma(\xi, \vartheta, w)$ is a permutation of the set $\{\xi, \vartheta, w\}$ and

(G5) $G(\xi, \vartheta, w) \leq G(\xi, z, z) + G(z, \vartheta, w)$ for all $\xi, \vartheta, w, z \in$ X . Then G is called a G - metric on X and the pair (X, G) is called a G - metric Space.

Also it is defined (see [79], Definition 4) that a G - metric space (X, G) is symmetric if

(G6) $G(\xi, \xi, \vartheta) = G(\xi, \vartheta, \vartheta)$ holds for all $\xi, \vartheta \in$ X.

1.1.35 Example: Let X $=\{a, b\}$. Define $G : X^3 \to [0, \infty)$ by G(a, a, a) = G(b, b, b) = 0; G(a, a, b) = 1, G(a, b, b) = 2 and extend G to all of X^3 by using (G4). Then (X, G) is a G-metric space. Also since G(a, a, b) $\neq$ G(a, b, b), the space (X, G) is not a symmetric G - metric space.

1.2 S-metric spaces

In 2012, Shaban Sedghi, Nabi Shobe and Abdelkrim Aliouche [79] defined S-metric spaces as follows:

1.2.1 Definition: [79] Let X be a non empty set. The mapping $S : X^3 \to [0, \infty)$ is said to be an S-metric on X, if it satisfies the following for $\xi, \vartheta, w, z \in$ X

(S1) $S(\xi, \vartheta, w) \geq 0$

(S2) $S(\xi, \vartheta, w) = 0$ if and only if $\xi = \vartheta = w$.

(S3) $S(\xi, \vartheta, w) \leq S(\xi, \xi, z) + S(\vartheta, \vartheta, z) + S(w, w, z)$

Also the pair (X, S) is called an S-metric space.

1.2.2 Example: Let $X = \mathbb{R}$ and $S : X^3 \to [0, \infty)$ be defined by $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$, then (X, S) is an S-metric space. In this space , note that

(1.2.3.) $S(1, 2, 3) \neq S(2, 3, 1)$,

since S(1,2,3) = 4 and S(2,3,1) = 2.

1.2.4 Example: Let X =$\mathbb{R}$ and $S : X^3 \to [0, \infty)$ be defined by $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for $\xi, \vartheta, w \in X$. Then (X, S) is an S-metric space in which

(1.2.5) $S(3, 3, 1) > S(3, 1, 2)$,

since S(3,3,1) = 4 and S(3,1,2) = 2

1.2.6 Remark: It was shown in ([79], Lemma 2.5) that $S(\xi, \xi, \vartheta) = S(\vartheta, \vartheta, \xi)$ for all $\xi, \vartheta \in X$

1.2.7 Example: [79] Define S:$X^3 \to [0, \infty)$ by $S(\xi, \vartheta, w) = d(\xi, \vartheta) + d(\xi, w) + d(\vartheta, w)$ for any $\xi, \vartheta, w \in$ X, where (X, d) be a metric space. Then (X, S) is an S-metric space.

1.2.8 Example: Suppose X=[0, 1] and S:$X^3 \to [0, \infty)$ be defined by

$$S(\xi, \vartheta, w) = \begin{cases} 0 & \text{if } \xi = \vartheta = w \\ max\{\xi, \vartheta, w\} & otherwise \end{cases}.$$

Then (X, S) is an S-metric space.

Observations on the three generalizations of metric

First we note that the three generalized metrics D*, G and S on any non empty set X are functions defined on X^3 into $[0, \infty)$.

1.2.9. The D*-metric space given in Example 1.1.31. is not a G - metric space since it does not possess the condition (G3) in view of (1.1.32.).

1.2.10 The G - metric space in Example 1.1.35. is not symmetric and therefore it does not possess the property mentioned in Remark 1.1.33. Hence it is not a D*- metric space.

1.2.11. The G -metric space given in Example 1.1.35. is not an S-metric space as it fails to possess the property mentioned in 1.2.6.

1.2.12. The S -metric space given in Example 1.2.4. is not a G-metric space since condition (G3) fails in view of (1.2.5.)

Thus the notions of D*-metric space and G - metric space are independent. Also the G - metric and S-metric are independent concepts on a nonempty set. However one can prove that every D*-metric space is an S-metric space but not conversely (Ex:1.2.2., in view of 1.2.5.).

Hereafter we consider, in this thesis, S-metric, S_b-metric and bicomplex metric on non empty sets and fixed point theorems on such spaces.

1.2.13 Definition: Let (X, S) be an S-metric space and A $\subset$ X

(i) For any $\xi \in X$, by S-ball about ξ, denoted by $B_s(\xi, r)$ we mean the set $\{\vartheta \in X : S(\vartheta, \vartheta, \xi) < r\}$ where $r > 0$

(ii) If for every $\xi \in$ A, there exists r $>$ 0 such that $B_s(\xi, r) \subset A$ then the subset A is called an open subset of X .

(iii) A subset A of X is said to be S-bounded if there exists M $>$ 0 such that $S(\xi, \xi, \vartheta) <$ M for all $\xi, \vartheta \in$ A.

It has been proved in [79] that $B_s(\xi, r)$ is an open set in X and that the topology τ generated by the open balls as a basis is called the topology induced by the

S-metric on X. If (X, τ) is a compact topological space, then we say (X, S) is a compact S -metric space.

1.2.14 Definition: Let (X, S) be an S-metric space. A sequence $\{\xi_n\}$ in X is said to be a

(i) convergent if there is a $\xi \in$ X such that $S(\xi_n, \xi_n, \xi) \to 0$ as n $\to \infty$; that is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all n$\geq n_0$, we have $S(\xi_n, \xi_n, \xi) < \epsilon$ and we write in this case that $lim_{n\to\infty}\xi_n = \xi$.

(ii) Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(\xi_n, \xi_n, \xi_m) < \epsilon$ for each n,m $\geq n_0$.

It is easy to see that (in fact proved in [79], Lemma 2.10 and Lemma 2.11) that, if $\{\xi_n\}$ converges to ξ in (X, S) then ξ is unique and $\{\xi_n\}$ is a Cauchy sequence in (X, S). However a Cauchy sequence in (X, S) need not be convergent as shown in the following example.

1.2.15 Example: Let X = (0, 1] and $S(\xi, \vartheta, w) = |\xi - \vartheta| + |\vartheta - w| + |w - \xi|$ for $\xi, \vartheta, w \in$ X. Then (X, S) is an S-metric space. Taking $\xi_n = \frac{1}{n}$ for n = 1,2,3,... then $S(\xi_n, \xi_n, \xi_m) = 2|\frac{1}{n} - \frac{1}{m}|$ so that $S(\xi_n, \xi_n, \xi_m) \to 0$ as n,m $\to \infty$ proving that $\{\xi_n\}$ is a Cauchy sequence in (X, S) but $\{\xi_n\}$ does not converge to any point in X.

1.2.16 Definition: An S-metric space (X, S) is said to be complete if every Cauchy sequence in X is converges to a point in X.

The S-metric space given in Example 1.2.15. is not complete.

1.2.17 Lemma: [80] In the S-metric space, we observe

(i) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(\vartheta, \vartheta, w)$ and

(ii) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(w, w, \vartheta)$

1.2.18 Definition: [79] Let (X, S) be an S-metric space. Then a mapping $h : X \to X$ is said to be an S-contraction if there exists a constant $0 \leq \tau < 1$ such that

$$S(h(\xi), h(\xi), h(\vartheta)) \leq \tau \ S(\xi, \xi, \vartheta) \quad for\ all \quad \xi, \vartheta \in X.$$

In 2015, F.Khajasteh, Satish Shukla and S.Radenovic [81] introduced simulation function and the concept of Z-contration in relation to simulation function and proved an fixed point theorem which generalizes the Banach contraction principle. Very recently, Murat Olgun, O.Bicer and T.Alyildiz [82] defined generalized Z-contraction in relation to the simulation function and proved a fixed point theorem.

In the year 2019, Nihal Tas, Nihal Ylimaz Ozgur and Nabil Mlaiki [83] proved an fixed point theorem by employing the collection of simulation mappings on S-metric spaces.

1.2.19 Definition: [81] We say that a mapping $\gamma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation mapping if:
(γ1) $\gamma(0, 0) = 0$
(γ2) $\gamma(p, q) < q - p$ for p,q > 0
(γ3) If $\{p_n\}, \{q_n\}$ are sequences of (0,∞) such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then $\lim_{n\to\infty}$sup $\gamma(p_n, q_n) < 0$.

We indicate Z as the collection of all simulation mappings. For example, $\gamma(p, q) = \tau q - p$ for 0$\leq$ τ <1 belonging to Z.

1.2.20 Definition: [81] Let (X, d) be a metric space and $\gamma \in Z$. Then a mapping $h : X \to X$ is said to be a Z-contraction in relation to γ if

$$\gamma(d(h\xi, h\vartheta), d(\xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in X.$$

By considering the Definition (1.2.20). It is concluded that each Banach contraction becomes Z-contraction in relation to $\gamma(p, q) = \tau q - p$ with $0 \leq \tau < 1$. Further, it can be established from the definition of the simulation mapping that $\gamma(p, q) < 0$ for each p $\geq q > 0$. Hence, assume that h is a Z-contraction in relation to $\gamma \in$ Z then

$$d(h\xi, h\vartheta) < d(\xi, \vartheta) \text{ for all distinct } \xi, \vartheta \in X.$$

1.2.21 Theorem: [81] In complete metric space (X, d), each Z-contraction has a unique fixed point and furthermore the fixed point is the limit of every Picard's sequence.

Nowadays, the study of fixed point theorems for self maps satisfying different contraction conditions is the center of rigorous research activities. In this direction, Dutta et al. [85] introduced (ψ, ϕ)-weakly contractive maps in 2008 and obtained some fixed point results for such contractions. Later, G.V.R. Babu et al. [86] introduced (ψ, ϕ)-almost weakly contractive maps in G-metric spaces in 2014. Fixed points of contractive maps on S-metric spaces were studied by several authors [87], [88] and [89]. Since then, several contractions have been considered for proving fixed point theorems.

1.2.22 Definition: Let (X, S) and (Y, S') be two S-metric spaces. Then a function h:X→Y is S-continuous at a point $\xi \in$X if it is S-sequentially continuous at ξ. That is, whenever $\{\xi_n\}$ is S-convergent to ξ, we have h(ξ_n) is S'-convergent to h(ξ).

1.2.23 Lemma: [84] Let (X, S) be an S-metric space and $\{\xi_n\}$ be a sequence in X such that $\lim_{n\to\infty} S(\xi_n, \xi_n, \xi_{n+1}) = 0$. If $\{\xi_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of natural numbers with $n_k > m_k > k$ such that $S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \geq \epsilon$, $S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) < \epsilon$ and
(i)$\lim_{k\to\infty} S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) = \epsilon$. (ii) $\lim_{k\to\infty} S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k-1}) = \epsilon$. (iv) $\lim_{k\to\infty} S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \epsilon$.

1.3 S_b-metric spaces

Recently, N.Mlaiki and N.Souayah [90] introduced the S_b-metric spaces as the generalization of b-metric spaces and S-metric spaces and proved some fixed point results were proved for such spaces in [90]. Very recently Ozgur and Tas [91] studied some relations between S_b-metric spaces and some other metric spaces. Some fixed point results in S_b-metric space were also studied by different authors in [[90]-[92]].

1.3.1 Definition: [90] Let $X \neq \emptyset$ and $s \geq 1$. Then we say a mapping $S_b : X^3 \to [0, \infty)$ is an S_b-metric on Ω if :
(i) $S_b(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$.
(ii)$S_b(\xi, \vartheta, w) \leq s[S_b(\xi, \xi, a) + S_b(\vartheta, \vartheta, a) + S_b(w, w, a)]$

$\forall \xi, \vartheta, w, a \in X$. The pair (X, S_b) is known as S_b-metric space.

Each S-metric space is a S_b-metric space for s=1, but the converse statement is not true. We find an example of S_b-metric, but not an S-metric on X in [91].

1.3.2 Definition: [86] Let (X, S_b) be an S_b-metric space for s≥1. Then S_b-metric is known as symmetric if $S_b(\xi, \xi, \vartheta) = S_b(\vartheta, \vartheta, \xi)$, $\forall \xi, \vartheta \in X$.

1.3.3 Lemma: [93] In S_b-metric space, we have $\forall \xi, \vartheta, w \in X$
(i) $S_b(\xi, \xi, \vartheta) \leq sS_b(\vartheta, \vartheta, \xi)$ and $S_b(\vartheta, \vartheta, \xi) \leq sS_b(\xi, \xi, \vartheta)$
(ii) $S_b(\xi, \xi, w) \leq 2sS_b(\xi, \xi, \vartheta) + s^2 S_b(\vartheta, \vartheta, w)$.

1.3.4 Definition: [90] Let (X, S_b) is an S_b-metric space and a sequence $\{\xi_n\}$ in X. Then
(i) $\{\xi_n\}$ is called an S_b-Cauchy sequence, if for every $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $S_b(\xi_n, \xi_n, \xi_m) \leq \epsilon$, $\forall n, m > n_0$.
(ii) $\{\xi_n\} \to \xi \iff$ for each $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $S_b(\xi_n, \xi_n, \xi) < \epsilon$ and $S_b(\xi, \xi, \xi_n) < \epsilon\ \forall n \geq n_0$ and we write as $\lim_{n\to\infty} \xi_n = \xi$.

1.3.5 Definition: [90] We say that (X, S_b) is complete if each S_b-Cauchy sequence is S_b-Convergent in X.

Tas and Ozgur [91] proved the following theorems in S_b-metric spaces.
1.3.6 Theorem: Let (X, S_b) be a complete S_b-metric space and s≥1. If h is a self map on X satisfy

$$S_b(h\xi, h\xi, h\vartheta) \leq c\ S_b(\xi, \xi, \vartheta)$$

$\forall \xi, \vartheta \in X$, where $0 < c < \frac{1}{s^2}$. Then h has a unique fixed point in X.

1.3.7 Example: [92] Let (X, S) be an S-metric space and $S_*(\xi, \vartheta, w) = [S(\xi, \vartheta, w)]^q$, where $q > 1$ is a real number.
Note that S_* is a S_b-metric with s = $2^{2(q-1)}$. Obviously, S_* satisfies conditions
(i) $0 < S_*(\xi, \vartheta, w)$, for all $\xi, \vartheta, w \in X$ with $\xi \neq \vartheta \neq w$.
(ii) $S_*(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$.
If 1 <q< ∞, then the convexity of the function $f(\xi) = \xi^q, (\xi > 0)$ implies that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$.

Thus, for each $\xi, \vartheta, w, a \in X$, we obtain,

$$\begin{aligned} S_*(\xi,\vartheta,w) &= S(\xi,\vartheta,w)^q \\ &\leq ([S(\xi,\xi,a)+S(\vartheta,\vartheta,a)]+S(w,w,a))^q \\ &\leq 2^{q-1}([S(\xi,\xi,a)+S(\vartheta,\vartheta,a)]^q+S(w,w,a)^q) \\ &\leq 2^{q-1}([2^{q-1}(S(\xi,\xi,a)^q+S(\vartheta,\vartheta,a)^q)]+2^{q-1}S(w,w,a)^q) \\ &\leq 2^{2(q-1)}(S(\xi,\xi,a)^q+S(\vartheta,\vartheta,a)^q+S(w,w,a)^q). \\ &\leq 2^{2(q-1)}(S_*(\xi,\xi,a)+S_*(\vartheta,\vartheta,a)+S_*(w,w,a)). \end{aligned}$$

So, S_* is a S_b-metric with s = $2^{2(q-1)}$.

In this article we indicate:

(i)$\Psi = \{\psi : [0,\infty) \to [0,\infty) : \psi$ is non decreasing, continuous and ψ(t)=0 $\iff$ t=0.$\}$

(ii) $\Phi = \{\phi : [0,\infty) \to [0,\infty)$: ϕ is continuous, $\phi(t) = 0 \iff t = 0\}$.

1.3.8 Lemma: [92] Let $\{\xi_n\}$ be S_b-convergent to ξ in S_b-metric space (X, S_b) for s≥1, then we obtain:

(i)$\frac{1}{2s}S_b(\vartheta,\vartheta,\xi) \leq \liminf_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq \limsup_{n\to\infty} S_b(\vartheta,\vartheta,\xi_n) \leq 2sS_b(\vartheta,\vartheta,\xi)$ and

(ii)$\frac{1}{s^2}S_b(\xi,\xi,\vartheta) \leq liminf_{n\to\infty}S_b(\xi_n,\xi_n,\vartheta) \leq \limsup_{n\to\infty} S_b(\xi_n,\xi_n,\vartheta) \leq s^2S_b(\xi,\xi,\vartheta)$.

1.3.9 Lemma: Let $\{\xi_n\}$ be a sequence in S_b-metric space (X, S_b) such that $\lim_{n\to\infty} S_b(\xi_n,\xi_n,\xi_{n+1}) = 0$.

If sequence $\{\xi_n\}$ is not Cauchy, then we find an $\epsilon > 0$ and $\{m_k\}$ and $\{n_k\}$ are sequences of natural numbers with $n_k > m_k > k$ so that

$S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) \geq \epsilon$, $S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) < \epsilon$ and

(i)$\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k}) = \epsilon$. (ii) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k}) = \epsilon$.

(iii) $\lim_{k\to\infty} S_b(\xi_{m_k},\xi_{m_k},\xi_{n_k-1}) = \epsilon$. (iv) $\lim_{k\to\infty} S_b(\xi_{m_k-1},\xi_{m_k-1},\xi_{n_k-1}) = \epsilon$.

1.4 Bicomplex valued metric spaces

Segre's [94] paper, published in 1892 made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology.

Price [95] developed the bicomplex algebra and function theory. In this field an impressive body of work has been developed by different researchers during the last few years. One can see some of the attempts in [96], [97], [98].

Azam et al. [99] introduced a concept of complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard and Imdad [100] generalized the result obtained by Azam et al. [99] and they proved another common fixed point theorem satisfying some rational inequality in complex valued metric space.

Choudhury et al. [[101],[102]] proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Also one can see the attempts in [103] and [104].

Rao et al. [105] introduced the concept of complex-valued b-metric spaces and proved a common fixed point theorem in complex valued b-metric spaces.

We denote $C_0 = \mathbb{R}$(Real numbers), $C_1 = \mathbb{C}$(Complex numbers) and C_2 as the set of bicomplex numbers.
Let z,w $\in C_1$ be any two complex numbers, then the partial order relation $\preceq$ on C_1 is defined as follows:
z $\preceq$ w if and only if Re(z)$\leq$ Re(w) and Im(z) $\leq$ Im(w).
Also z $\prec$ w if Re(z) $<$ Re(w) and Im(z) $<$ Im(w).
Segre's [94] defined the bicomplex number as:

$$\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2,$$

where $b_1, b_2, b_3, b_4 \in C_0$ and i_1, i_2 are the independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$,
we defined C_2 as:

$$C_2 = \{\zeta : \zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2, b_1, b_2, b_3, b_4 \in C_0\},$$

i.e.,

$$C_2 = \{\zeta : \zeta = z + i_2 w, z, w \in C_1\}$$

where $z = b_1 + b_2 i_1 \in C_1$ and $w = b_3 + b_4 i_1 \in C_1$.
If $\zeta = z+i_2w$ and $\gamma = u+i_2v$ then $\zeta \pm \gamma = (z+i_2w)\pm(u+i_2v) = (z\pm u)+i_2(w\pm v)$ and the product is $\zeta.\gamma = (z + i_2w).(u + i_2v) = (zu - wv) + i_2(zv + wu)$.

The norm $\|.\|: C_2 \to \mathbb{C}_0^{+}$is
defined by
$\|\zeta\| = \|z + i_2w\| =\{|z|^2 + |w|^2\}^{\frac{1}{2}} = (b_1^2 + b_2^2 + b_3^2 + b_4^2)^{\frac{1}{2}}$
where $\zeta = b_1 + b_2i_1 + b_3i_2 + b_4i_1i_2 = z + i_2w \in C_2$

The partial order relation $\preceq_{i_2}$ on C_2 is defined as:
Let $\zeta = z + i_2w$, $\gamma = u + i_2v \in C_2$ then
$\zeta \preceq_{i_2} \gamma$ if and only if z $\preceq$ u and $w \preceq$ v.
i.e., $\zeta \preceq_{i_2} \gamma$ if :
(1) z = u, w = v or
(2) z $\prec$ u, w = v or
(3) z = u, w $\prec$ v or
(4) z $\prec$ u, w $\prec$ v.

For any two bicomplex numbers $\zeta, \gamma \in C_2$:
(i) $\zeta \preceq_{i_2} \gamma \implies \|\zeta\| \leq \|\gamma\|$
(ii) $\|\zeta + \gamma\| \leq \|\zeta\| + \|\gamma\|$

1.4.1 Definition: [106] Let X be a nonempty set. Then the mapping $d : X \times X \to C_2$ is said to be bicomplex-valued metric on X if it satisfies the following conditions:
(1) $0 \preceq_{i_2} d(z, w)$ for all z,w $\in$ X,
(2) $d(z, w) = 0$ if and only if z = w,
(3) $d(z, w) = d(w, z)$ for all z,w $\in$ X and
(4) $d(z, w) \preceq_{i_2} d(z, u) + d(u, w)$ for all z,w,u $\in$ X.
Then (X, d) is called the bicomplex valued metric space.

Let (X, d) be a bicomplex valued metric space. Then

1.4.2 Definition: [106]

(1) we say that a sequence $\{w_n\}$ in X converges to a point w if for any $0 \prec_{i_2}$ r $\in C_2$ there exists $n_0 \in \mathbb{N}$ such that $d(w_n, w) \prec_{i_2}$ r, for all $n > n_0$ and we write $\lim_{n\to\infty} w_n = w$.

(2) we say that a sequence $\{w_n\}$ in X is a Cauchy sequence if for any $0 \prec_{i_2}$ r $\in C_2$ there exists $n_0 \in \mathbb{N}$ such that $d(w_n, w_{n+m}) \prec_{i_2}$ r, for all m,n $\in \mathbb{N}$ and $n > n_0$.

(3) we say that (X, d) is complete bicomplex valued metric space if every Cauchy sequence in X is convergent in X.

1.4.3 Definition: We say that two maps h, k : X $\to$ X are commutes if hk(z) = kh(z) for all z $\in$ X.

1.4.4 Definition: We say that two maps h, k : X $\to$ X are compatible mappings if $\lim_{n\to\infty} d(hkz_n, khz_n) = 0$ whenever $\{z_n\}$ be any sequence in X such that $\lim_{n\to\infty} hz_n = \lim_{n\to\infty} kz_n =$ z for some z $\in$ X.

In 1998, Jungck and Rhoades [112] introduced the concept of weakly compatible mappings and proved fixed point theorems using these mappings on metric spaces.

1.4.5 Definition: We say that two maps h, k : X $\to$ X are weakly compatible if hz = kz for some z $\in$ X implies hk(z) = kh(z).

1.4.6 Definition: Let h,k,A,B : X $\to$ X are four maps. We say that $\{h, A\}$ and $\{k, B\}$ are satisfy the CLR_{AB} property if there exists two sequences $\{z_n\}$ and $\{w_n\}$ in X such that $\lim_{n\to\infty} hz_n = \lim_{n\to\infty} Az_n = \lim_{n\to\infty} kw_n = \lim_{n\to\infty} Bw_n =$ z for some z $\in$ A(X)$\cap$B(X).

In 1996, S.S.Chang and et al. [113] introduced the coupled fixed point as follows.

1.4.7 Definition: An element $(\xi, \vartheta) \in X \times X$ is called a coupled fixed point of the mapping h : $X \times X \to$ X if $h(\xi, \vartheta) = \xi$ and $h(\vartheta, \xi) = \vartheta$.

1.4.8 Definition: [107] Let $\{z_n\}$ be any sequence in (X, d). Then we say that $\{z_n\}$ is converges to a point z if and only if $\lim_{n\to\infty} \|d(z_n, z)\| = 0$.

1.5 A brief summary

The main work in this thesis centers around the development of various contractions for self maps on S-metric spaces, S_b-metric spaces and bicomplex valued metric spaces. We investigate existence of fixed and common fixed points of maps for such contractions. Our results extend, unify and generalize several known results as well. The work of this thesis is organized in six chapters.

The first chapter is introductory and present a background material needed for the rest of the chapters.

In second chapter, we define an (ψ, ϕ) - almost weakly generalized contractive map in S-metric spaces and prove the following theorem for an existence and uniqueness of fixed point of such maps. Furthermore we deduce some results as corollaries to our result and provide an example to validate our result.

1.5.1 Definition: Let (X, S) be an S-metric space. A map h:X $\rightarrow$ X is called (ψ, ϕ) - almost weakly generalized contractive if it satisfies the inequality

$$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w)$$

for all $\xi, \vartheta, w \in$ X, $\psi \in \Psi$, $\phi \in \Phi$ and L ≥ 0, where
$M(\xi, \vartheta, w) = max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}$,
$\theta(\xi, \vartheta, w) = min\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, h\xi), S(\xi, \xi, hw)\}$.

1.5.2 Theorem: Let (X, S) be a complete S-metric space and h: X $\rightarrow$ X be a (ψ, ϕ) - almost weakly generalized contractive mapping. Then h has a unique fixed point in X.

The intent of the third chapter is to present the following fixed point theorem by defining generalized Z_s-contractions in relation to the simulation function in S-metric space. In addition to that, we bestow an example which supports our results.

1.5.3 Lemma: If $h : X \to X$ is a generalized Z_s-contraction in relation to γ, then h is an asymptotically regular at each point $\xi \in X$.

1.5.4 Lemma: If h is a generalized Z_s-contraction in relation to γ, then the Picard sequence $\{\xi_n\}$ generated by h such that h$\xi_{n-1} = \xi_n$, to each n∈N with initial value $\xi_0 \in X$ is a bounded sequence.

1.5.5 Theorem: Let (X, S) be a complete S-metric space and $h : X \to X$ be a self-mapping. If h is a generalized Z_s-contraction in relation to γ, then h has a unique fixed point $\eta \in$ X and the Picard sequence $\{\xi_n\}$ converges to the fixed point η.

In Chapter - IV of the thesis, we define (ψ, ϕ) - weakly generalized contractive map in S_b-metric spaces and prove the following theorem for an existence and uniqueness of fixed point. We also give an example to support of our result.

1.5.6 Definition: Let (X, S_b) be an S_b-metric space for s≥1. Let h be a self map of X. Then we say h is a (ψ, ϕ) - weakly generalized contractive map if ∃ L≥0, $\psi \in \Psi$ and $\phi \in \Phi$ such that
$\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w)$
where P$(\xi, \vartheta, w) = \max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$
$\frac{1}{4s^2}[S_b(h\xi, h\vartheta, hw) + S_b(h\xi, h\xi, \xi) S_b(h\xi, h\xi, w) S_b(hw, hw, \vartheta)]\}$
and Q$(\xi, \vartheta, w) = \min\{S_b(hw, \xi, \xi), S_b(h\xi, \vartheta, \vartheta), S_b(h\xi, w, w), S_b(h\xi, \vartheta, w)\}$
$\forall \xi, \vartheta, w \in X$.

1.5.7 Theorem: Let h be a self map on a complete symmetric S_b-metric space (X, S_b) for s≥1. If h is a (ψ, ϕ) - weakly generalized contractive map, then h has a unique fixed point in X.

In Chapter - V of the thesis, we establish the following fixed point and common fixed-point theorems in S_b-metric spaces using implicit relation. The results presented in this paper extend and generalize several results from the existing literature.

1.5.8 Definition(Implicit Relation): Let Ψ be the family of all real valued continuous functions ψ: $\mathbb{R}_+^5 \to \mathbb{R}_+$ non-decreasing in the first argument for five variables. For some $q \in [0, \frac{1}{s^2}]$, where s $\geq$1, we consider the following conditions.
(R1) For $\xi, \vartheta \in \mathbb{R}_+$, if $\xi \leq \psi(\vartheta, s\xi, s\vartheta, s\xi, \xi + s\vartheta)$ then $\xi \leq q\vartheta$.
(R2) For $\xi, \vartheta \in \mathbb{R}_+$, if $\xi \leq \psi(0, 0, \xi, 0, 0)$ then $\xi = 0$.
(R3) For $\xi \in \mathbb{R}_+$, if $\xi \leq \psi(\xi, 0, 0, 0, \frac{\xi}{2})$ then $\xi = 0$.

1.5.9 Theorem: Let T be a self map on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$S_b(T\xi, T\vartheta, Tw) \leq \psi(S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T\xi), S_b(w, w, Tw), S_b(\xi, \xi, T\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T\vartheta) + S_b(w, w, T\xi)])$$

for all $\xi, \vartheta, w \in X$ and $\psi \in \Psi$. If ψ satisfies the conditions (R1), (R2) and (R3), then T has a unique fixed point in X.

1.5.10 Theorem: Let T_1 and T_2 be two self maps on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$S_b(T_1\xi, T_1\vartheta, T_2w) \leq \psi(S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T_1\xi), S_b(w, w, T_2w), S_b(\xi, \xi, T_1\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1\vartheta) + S_b(w, w, T_1\xi)])$$

for all $\xi, \vartheta, w \in X$ and $\psi \in \Psi$. If ψ satisfies the conditions (R1), (R2) and (R3), then T_1 and T_2 have a unique common fixed point in X.

1.5.11 Theorem: Let T_1 and T_2 be two continuous self maps on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$S_b(T_1^p\xi, T_1^p\vartheta, T_2^qw) \leq \psi(S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T_1^p\xi), S_b(w, w, T_2^qw), S_b(\xi, \xi, T_1^p\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1^p\vartheta) + S_b(w, w, T_1^p\xi)])$$

for all $\xi, \vartheta, w \in X$, where p and q are integers and $\psi \in \Psi$. If ψ satisfies the conditions (R1), (R2) and (R3), then T_1 and T_2 have a unique common fixed point in X.

1.5.12 Theorem: Let $\{G_\alpha\}$ be a family of continuous self maps on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$S_b(G_\alpha\xi, G_\alpha\vartheta, G_\beta w) \leq \psi(S_b(\xi,\vartheta,w), S_b(\vartheta,\vartheta,G_\alpha\xi), S_b(w,w,G_\beta w), \\ S_b(\xi,\xi,G_\alpha\vartheta), \frac{1}{2s}[S_b(\vartheta,\vartheta,G_\alpha\vartheta) + S_b(w,w,G_\alpha\xi)])$$

for all $\xi, \vartheta, w \in X$ and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha \neq \beta$. Then there exists a unique $\eta \in X$ satisfying $G_\alpha\eta = \eta$, for all $\alpha \in \Psi$.

In Chapter VI of the thesis, we establish the following two unique common fixed point theorems for four self-mappings and six self-mappings and a common coupled fixed point theorem in the bicomplex valued metric space. In the first theorem, we establish a common fixed point theorem for four self-mappings by using weaker conditions such as weakly compatibility, generalized contraction and CLR_{AB} property. Then, in the Second theorem, we establish a common fixed point theorem for six self-mappings with the help of weakly compatibility and inclusion relations by using the generalized contraction. Further, in the third theorem, we establish a common coupled fixed point theorem using a different contraction in the bicomplex valued metric space.

1.5.13 Theorem: Let (X, d) be a complete bicomplex valued metric space and h,k,A and B are self mappings on X satisfying
(i) $d(h\varpi, k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi, B\vartheta) + \tau_2 d(A\varpi, h\varpi) + \tau_3 d(B\vartheta, k\vartheta)$, $\forall \varpi, \vartheta \in X$,
where τ_1,τ_2 and τ_3 be non negative real number such that $\tau_1 + \tau_2 + \tau_3 < 1$.
(ii) $\{B, k\}$ and $\{A, h\}$ be weakly compatible,
(iii) $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property.
Then h,k,A and B have a unique common fixed point in X.

1.5.14 Theorem: Let (X, d) be a complete bicomplex valued metric space and H,I,C,P,Q,R be the self mappings on X satisfies (i) H(X) $\supseteq$ QR(X) and I(X) $\supseteq$ CP(X) (ii) $d(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1 d(H\varpi, I\vartheta) + \tau_2 d(H\varpi, CP\varpi) + \tau_3 d(I\vartheta, QR\vartheta)$ + $\tau_4 d(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in X$, where τ_1,τ_2,τ_3 and τ_4 be non negative real number such that $\tau_1+\tau_2+\tau_3+2\tau_4 < 1$. (iii) Suppose (QR,I) and (CP,H) are weakly compatible and (iv) (Q,R), (Q,I), (R,I),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P,I and H have a unique common fixed point in X.

1.5.15 Theorem: Let (X, d) be a complete bicomplex valued metric space and h,k: X × X → X be two functions satisfy
$d(h(\varpi, \jmath), k(\rho, \sigma)) \preceq_{i_2} \tau_1 \, \frac{d(\varpi,\rho)+d(\jmath,\sigma)}{2} + \tau_2 \, \frac{d(\varpi,h(\varpi,\jmath))+d(\rho,\varpi)}{2} + \tau_3 \, \frac{d(\varpi,h(\varpi,\jmath))+d(\rho,k(\rho,\sigma))}{2}$
where $\varpi, \jmath, \rho, \sigma \in X$ and τ_1, τ_2 and τ_3 are non negative integers such that $\tau_1 + \tau_2 + \tau_3 < 1$. Then h and k have a unique common coupled fixed point in X × X.

Chapter 2

Fixed point results for (ψ, ϕ)-generalized almost weakly contractive maps in S-metric spaces

2.1 Introduction:

A (ψ, ϕ)- almost weakly generalized contractive map is defined in this chapter. The existence and uniqueness of the fixed point of such maps are demonstrated in S-metric spaces. Also we deduce some existing results as special cases of our result. Moreover, we give an example in support of the results.

Dutta et al. [85] introduced (ψ, ϕ) - weakly contractive maps in 2008 and obtained some fixed point results for such contractions in metric spaces. Later, for weakly contractive maps in G-metric spaces, Al-Sharif, Khandaqji and Al-Khaleel [111] established the following theorem in 2012.

2.1.1 Theorem: [111] Let h be a self map on a complete G-metric space (X, G). If $\psi \in \Psi$ and $\phi \in \Phi$ so that

$$\begin{aligned}\psi(G(h\xi, h\vartheta, hw)) \leq \psi(max\{&G(\xi, \vartheta, w), G(\xi, h\xi, h\xi), G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw),\\ &\alpha G(h\xi, h\xi, \vartheta) + (1-\alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi)\\ &+ (1-\beta)G(\vartheta, h\vartheta, h\vartheta)\}) - \phi(max\{G(\xi, \vartheta, w), G(\xi, h\xi, h\xi),\\ &G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw), \alpha G(h\xi, h\xi, \vartheta)\\ &+ (1-\alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi) + (1-\beta)G(\vartheta, h\vartheta, h\vartheta)\})\end{aligned}$$

for all $\xi, \vartheta, w \in$X, here $\alpha, \beta \in (0, 1)$, then h has one and only one fixed point, say $u \in X$ and G-continuous at u.
We now provide the following (ψ, ϕ)- almost weakly generalized contractive map in S-metric spaces along with an illustration.

2.1.2 Definition: Consider a self map h on an S-metric space (X, S). We say that h is (ψ, ϕ) - almost weakly generalized contractive map if

$$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w) \quad (2.1.1.)$$

for all $\xi, \vartheta, w \in$ X, $\psi \in \Psi$, $\phi \in \Phi$ and L ≥ 0, where
$M(\xi, \vartheta, w) = max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}$,
$\theta(\xi, \vartheta, w) = min\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, h\xi), S(\xi, \xi, hw)\}$.

2.1.3 Example: Let X = $[0, \frac{8}{7}]$ and define h : X $\to$ X by

$$h\xi = \begin{cases} \frac{\xi}{10} & \text{when } \xi \in [0,1] \\ \xi - \frac{4}{5} & \text{when } \xi \in (1, \frac{8}{7}] \end{cases}.$$

We define S: $X^3 \to [0, \infty)$ by S$(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for all $\xi, \vartheta, w \in$ X.

Clearly (X, S) is definitely a complete S-metric space.

Functions $\psi, \phi : [0, \infty) \to [0, \infty)$ are defined by

$$\psi(\upsilon) = \upsilon, \forall \upsilon \geq 0 \text{ and } \phi(\upsilon) = \begin{cases} \frac{\upsilon}{2} & \text{when } \upsilon \in [0,1] \\ \frac{\upsilon}{\upsilon+1} & \text{when } \upsilon \geq 1. \end{cases}.$$

Now, verify that h holds the inequality (2.1.1.).

Case(i): Let $\xi, \vartheta, w \in$ [0,1].

We suppose that $\xi > \vartheta > w$, w.l.o.g.,

$S(h\xi, h\vartheta, hw) = S(\frac{\xi}{10}, \frac{\vartheta}{10}, \frac{w}{10}) = \frac{1}{10}(|\xi - w| + |\vartheta - w|)$ and

$S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$.

Subcase (a): If $|\xi - w| + |\vartheta - w| \in$ [0, 1].

In this case,

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= \frac{1}{10}(|\xi - w| + |\vartheta - w|) \leq \frac{1}{2}(|\xi - w| + |\vartheta - w|) \\ &= \frac{1}{2}S(\xi, \vartheta, w) \leq \frac{1}{2}M(\xi, \vartheta, w) \\ &= M(\xi, \vartheta, w) - \frac{1}{2}M(\xi, \vartheta, w) \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\ &\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w). \end{aligned}$$

Subcase(b): If $|\xi - \vartheta| + |\vartheta - w| \geq$1.

In this case,

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= \frac{1}{10}(|\xi - \vartheta| + |\vartheta - w|) \leq |\xi - \vartheta| + |\vartheta - w| - \frac{|\xi - \vartheta| + |\vartheta - w|}{1 + |\xi - \vartheta| + |\vartheta - w|} \\ &= S(\xi, \vartheta, w) - \frac{S(\xi, \vartheta, w)}{1 + S(\xi, \vartheta, w)} \\ &= \frac{(S(\xi, \vartheta, w))^2}{1 + S(\xi, \vartheta, w)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\ &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\ &\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w). \end{aligned}$$

Case(ii): Let $\xi, \vartheta, w \in (1, \frac{8}{7}]$.

We suppose that $\xi > \vartheta > w$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\xi - \frac{4}{5}, \vartheta - \frac{4}{5}, w - \frac{4}{5}) = |\xi - w| + |\vartheta - w| \\
&\leq \frac{2}{7} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
&= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case(iii): Let $\vartheta, w \in [0,\ 1]$ and $\xi \in (1,\ \frac{8}{7}]$.

We suppose that $\vartheta\ >\ $ w, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\xi - \frac{4}{5}, \frac{\vartheta}{10}, \frac{w}{10}) = |\xi - \frac{4}{5} - \frac{w}{10}| + |\frac{\vartheta}{10} - \frac{w}{10}| \\
&= \xi - \frac{w}{10} - \frac{4}{5} + \frac{\vartheta - w}{10} = \xi + \frac{\vartheta}{10} - \frac{w}{5} - \frac{4}{5} \\
&= \frac{31}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, f\xi)} \\
&= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case(iv): Let w$\in [0,\ 1]$ and $\xi, \vartheta \in (1,\ \frac{8}{7}]$.

We suppose that $\xi > \vartheta$, w.l.o.g.,

$$S(h\xi, h\vartheta, hw) = S(\xi - \frac{4}{5}, \vartheta - \frac{4}{5}, \frac{w}{10}) = |\xi - \frac{4}{5} - \frac{w}{10}| + |\vartheta - \frac{4}{5} - \frac{w}{10}|$$

$$\begin{aligned}
&= \xi + \vartheta - \frac{w}{5} - \frac{8}{5} = \frac{12}{35} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
&= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
&= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case (v): Let $\xi, \vartheta \in [0,\ 1]$ and $w \in (1,\ \frac{8}{7}]$.

We suppose that $\xi > \vartheta$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \frac{\vartheta}{10}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\frac{\vartheta}{10} - w + \frac{4}{5}| \\
&= |\frac{4}{5} - (w - \frac{\xi}{10})| + |\frac{4}{5} - (w - \frac{\vartheta}{10})| = w - \frac{\xi}{10} - \frac{4}{5} + w - \frac{\vartheta}{10} - \frac{4}{5} \\
&= 2w - \frac{\xi + \vartheta}{10} - \frac{8}{5} = \frac{41}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
&= S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
&= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case (vi): Let $\xi \in [0,\ 1]$ and $w, \vartheta \in (1,\ \frac{8}{7}]$.

We suppose that $w > \vartheta$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \vartheta - \frac{4}{5}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\vartheta - w| \\
&= w - \frac{\xi}{10} - \frac{4}{5} + w - \vartheta = 2w - \frac{\xi}{10} - \frac{4}{5} - \vartheta \\
&\leq \frac{27}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
&= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
&= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

We showed that h is a (ψ, ϕ) - almost weakly generalized contractive map on X from all the cases mentioned above.

2.1.4 Lemma: [84] Let $\{\xi_\ell\}$ be a sequence in an S-metric space (X, S) so that $\lim_{\ell\to\infty} S(\xi_\ell, \xi_\ell, \xi_{\ell+1}) = 0$. If $\{\xi_\ell\}$ is not a Cauchy, we can find an $\epsilon > 0$ and two sequences $\{m_\sigma\}$ and $\{\ell_\sigma\}$ of natural numbers with $\sigma < m_\sigma < \ell_\sigma$ so that $S(\xi_{m_\sigma}, \xi_{m_\sigma}, \xi_{\ell_\sigma}) \geq \epsilon$, $S(\xi_{m_\sigma-1}, \xi_{m_\sigma-1}, \xi_{\ell_\sigma}) < \epsilon$ and

(i)$\lim_{\sigma\to\infty} S(\xi_{m_\sigma}, \xi_{m_\sigma}, \xi_{\ell_\sigma}) = \epsilon$. (ii) $\lim_{\sigma\to\infty} S(\xi_{m_\sigma-1}, \xi_{m_\sigma-1}, \xi_{\ell_\sigma}) = \epsilon$.

(iii) $\lim_{\sigma\to\infty} S(\xi_{m_\sigma}, \xi_{m_\sigma}, \xi_{\ell_\sigma-1}) = \epsilon$. (iv) $\lim_{\sigma\to\infty} S(\xi_{m_\sigma-1}, \xi_{m_\sigma-1}, \xi_{\ell_\sigma-1}) = \epsilon$.

2.2 Main Results and Examples

Through this section we establish a fixed point theorem using the (ψ, ϕ) - almost weakly generalized contractive maps. Further, we derive some corollaries and bestow example to our result.

2.2.1 Theorem: Let h:$X \to X$ be a function on a complete S-metric space (X, S) and be a (ψ, ϕ) - almost weakly generalized contractive map. Then h has one and only one fixed point.

Proof: Consider an arbitrary $\xi_0 \in$ X. We establish a sequence $\{\xi_\ell\}$ such that h$\xi_\ell = \xi_{\ell+1}$, for $\ell =$ 0,1,2,....

If for some $\ell \in \mathbb{N}$, $\xi_\ell = \xi_{\ell+1}$, then ξ_ℓ is a fixed point of h.

Suppose $\xi_\ell \neq \xi_{\ell+1}$, for all $\ell \in \mathbb{N}$.

Consider,

$$\begin{aligned}
\psi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell)) &= \psi(S(h\xi_\ell, h\xi_\ell, h\xi_{\ell-1})) \\
&\leq \psi(max\{S(\xi_\ell, \xi_\ell, \xi_{\ell-1}), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell), \\
&\frac{1}{2}[S(\xi_\ell, \xi_\ell, h\xi_\ell) + S(\xi_\ell, \xi_\ell, h\xi_\ell)]\}) \\
&- \phi(max\{S(\xi_\ell, \xi_\ell, \xi_{\ell-1}), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell), \\
&\frac{1}{2}[S(\xi_\ell, \xi_\ell, h\xi_\ell) + S(\xi_\ell, \xi_\ell, h\xi_\ell)]\}) \\
&+ L.min\{S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_{\ell-1}, \xi_{\ell-1}, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_{\ell-1})\} \\
&= \psi(max\{S(\xi_\ell, \xi_\ell, \xi_{\ell-1}), S(\xi_\ell, \xi_\ell, \xi_{\ell+1})\}) - \phi(max\{S(\xi_\ell, \xi_\ell, \xi_{\ell-1}), \\
&S(\xi_\ell, \xi_\ell, \xi_{\ell+1})\}) + L.0
\end{aligned}$$

If max$\{S(\xi_\ell, \xi_\ell, \xi_{\ell+1}), S(\xi_\ell, \xi_\ell, \xi_{\ell-1})\} = S(\xi_\ell, \xi_\ell, \xi_{\ell+1})$, then we get

$$\psi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell)) \leq \psi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell)) - \phi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell))$$

that is, $\phi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell)) \leq 0$, which implies that $S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell) = 0$. Then we get $\xi_{\ell+1} = \xi_\ell$, which is a contradiction to our assumption that $\xi_\ell \neq \xi_{\ell+1}$, for each ℓ.

Therefore, max$\{S(\xi_\ell, \xi_\ell, \xi_{\ell+1}), S(\xi_\ell, \xi_\ell, \xi_{\ell-1})\} = S(\xi_\ell, \xi_\ell, \xi_{\ell-1})$,

and so we get

$$\psi(S(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell)) \leq \psi(S(\xi_\ell, \xi_\ell, \xi_{\ell-1})) - \phi(S(\xi_\ell, \xi_\ell, \xi_{\ell-1})) \qquad (2.2.1.)$$

that is, $\psi(S(\xi_{\ell+1},\xi_{\ell+1},\xi_\ell)) \le \psi(S(\xi_\ell,\xi_\ell,\xi_{\ell-1}))$.
Therefore we get, $S(\xi_{\ell+1},\xi_{\ell+1},\xi_\ell) \le S(\xi_\ell,\xi_\ell,\xi_{\ell-1})$, for all ℓ and the sequence $\{S(\xi_{\ell+1},\xi_{\ell+1},\xi_\ell)\}$ is bounded and decreasing. So, we can find p ≥ 0 so that

$$\lim_{\ell\to\infty} S(\xi_{\ell+1},\xi_{\ell+1},\xi_\ell) = p.$$

Letting $\ell \to \infty$ in equation (2.2.1.), we get

$$\psi(p) \le \psi(p) - \phi(p),$$

This, unless p = 0, is a contradiction.
Hence,

$$\lim_{\ell\to\infty} S(\xi_{\ell+1},\xi_{\ell+1},\xi_\ell) = 0. \qquad (2.2.2.)$$

Now, we claim that $\{\xi_\ell\}$ is a Cauchy sequence. Suppose not, we can have a $\epsilon > 0$ to which we can discover increasing sequences of integers $\{m_\sigma\}$ and $\{\ell_\sigma\}$ and sub sequences $\{\xi_{m(\sigma)}\}$ and $\{\xi_{\ell(\sigma)}\}$ of $\{\xi_\ell\}$ so that $\ell(\sigma)$ is the least index to which $\ell(\sigma) > m(\sigma) > \sigma$,

$$S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)}) \ge \epsilon \qquad (2.2.3.)$$

Then, we have

$$S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)-1}) < \epsilon \qquad (2.2.4.)$$

Now,

$$\begin{aligned}\epsilon \le S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)}) &= S(\xi_{\ell(\sigma)},\xi_{\ell(\sigma)},\xi_{m(\sigma)})\\ &\le 2S(\xi_{\ell(\sigma)},\xi_{\ell(\sigma)},\xi_{\ell(\sigma)-1}) + S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)-1})\\ &\le \epsilon + 2S(\xi_{\ell(\sigma)},\xi_{\ell(\sigma)},\xi_{\ell(\sigma)-1}) \;\; (Using\ equation\ (2.2.4))\end{aligned}$$

Letting $\sigma \to \infty$, we get

$$\lim_{\sigma\to\infty} S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)}) = \epsilon. \qquad (2.2.5.)$$

Also,

$$\begin{aligned}S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)}) &\le 2S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{m(\sigma)-1}) + S(\xi_{\ell(\sigma)},\xi_{\ell(\sigma)},\xi_{m(\sigma)-1})\\ &\le 2S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{m(\sigma)-1}) + 2S(\xi_{\ell(\sigma)},\xi_{\ell(\sigma)},\xi_{\ell(\sigma)-1})\\ &+ S(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}) \qquad (2.2.6.)\end{aligned}$$

and

$$\begin{aligned}S(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}) &\le 2S(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{m(\sigma)}) + S(\xi_{\ell(\sigma)-1},\xi_{\ell(\sigma)-1},\xi_{m(\sigma)})\\ &= 2S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{m(\sigma)-1}) + S(\xi_{m(\sigma)},\xi_{m(\sigma)},\xi_{\ell(\sigma)-1}) \qquad (2.2.7.)\end{aligned}$$

Taking $\sigma \to \infty$ in equation (2.2.7.) and utilizing equations (2.2.2.), (2.2.4.), (2.2.5.) and (2.2.6.)
we get

$$\lim_{\sigma\to\infty} S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}) = \epsilon \qquad (2.2.8.)$$

Setting $\xi = \xi_{m(\sigma)-1}, \vartheta = \xi_{m(\sigma)-1}$ and $w = \xi_{\ell(\sigma)-1}$ in equation (2.1.1.), we get

$$\begin{aligned}\psi(\epsilon) &\le \psi(S(\xi_{m(\sigma)}, \xi_{m(\sigma)}, \xi_{\ell(\sigma)})) = \psi(S(h\xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}, h\xi_{\ell(\sigma)-1}))\\
&\le \psi(max\{S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}), S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}),\\
&S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}), \frac{1}{2}[S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1})\\
&+ S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1})]\})\\
&- \phi(max\{S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}), S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}),\\
&S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}), \frac{1}{2}[S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1})\\
&+ S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1})]\})\\
&+ L.min\{S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}), S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{m(\sigma)-1}),\\
&S(\xi_{\ell(\sigma)-1}, \xi_{\ell(\sigma)-1}, h\xi_{m(\sigma)-1}), S(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, h\xi_{\ell(\sigma)-1})\}\end{aligned}$$

Taking $\sigma \to \infty$ and utilizing equation (2.2.8.), we obtain

$$\begin{aligned}&\psi(\epsilon) \le \psi(max\{\epsilon, 0, 0, 0\}) - \phi(max\{\epsilon, 0, 0, 0\}) + L.min\{0, 0, 0, \epsilon\}\\
&\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon) + L.0\end{aligned}$$

As $\epsilon > 0$, this will become contradiction. This proves that $\{\xi_\ell\}$ becomes a Cauchy in X and since X is complete, we can find a $\tau \in$ X so that $\{\xi_\ell\} \to \tau$ as $\ell \to \infty$.
Now we prove that h$\tau = \tau$.
Put $\xi = \xi_\ell, \vartheta = \xi_\ell$ and $w = \tau$ in equation (2.1.1.), then we have

$$\begin{aligned}\psi(S(\xi_{\ell+1}, \xi_{\ell+1}, h\tau)) &= \psi(S(h\xi_\ell, h\xi_\ell, h\tau))\\
&\le \psi(max\{S(\xi_\ell, \xi_\ell, \tau), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell),\\
&\frac{1}{2}[S(\xi_\ell, \xi_\ell, h\xi_\ell) + S(\xi_\ell, \xi_\ell, h\xi_\ell)]\})\\
&- \phi(max\{S(\xi_\ell, \xi_\ell, \tau), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell),\\
&\frac{1}{2}[S(\xi_\ell, \xi_\ell, h\xi_\ell) + S(\xi_\ell, \xi_\ell, h\xi_\ell)]\})\\
&+ L.min\{S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\xi_\ell), S(\tau, \tau, h\xi_\ell), S(\xi_\ell, \xi_\ell, h\tau)\}\end{aligned}$$

Letting $\ell \to \infty$, we get
$\psi(S(\tau, \tau, h\tau)) \le \psi(S(\tau, \tau, \tau)) - \phi(S(\tau, \tau, \tau)$+L.0

$\psi(S(\tau,\tau,h\tau)) \leq 0$. So, we get S$(\tau,\tau,h\tau)$=0.
Therefore h$\tau = \tau$. That is τ is a fixed point of h.
To show τ is unique, consider ℓ be another fixed point of h.
Using equation (2.1.1.), we consider

$$\begin{aligned}
\psi(S(\tau,\tau,\ell)) &= \psi(S(h\tau,h\tau,h\ell)) \\
&\leq \psi(max\{S(\tau,\tau,\ell), S(\tau,\tau,h\tau), S(\tau,\tau,h\tau), \frac{1}{2}[S(\tau,\tau,h\tau) + S(\tau,\tau,h\tau)]\}) \\
&- \phi(max\{S(\tau,\tau,\ell), S(\tau,\tau,h\tau), S(\tau,\tau,h\tau), \frac{1}{2}[S(\tau,\tau,h\tau) + S(\tau,\tau,h\tau)]\}) \\
&+ L.min\{S(\tau,\tau,h\tau), S(\tau,\tau,h\tau), S(\ell,\ell,h\tau), S(\tau,\tau,h\ell)\} \\
&That\ is,\quad \psi(S(\tau,\tau,\ell)) \leq \psi(S(\tau,\tau,\ell)) - \phi(S(\tau,\tau,\ell)),
\end{aligned}$$

a contradiction, unless S$(\tau,\tau,\ell) = 0$. Hence we get $\tau = \ell$.
Hence h has one and only one fixed point τ in X.

In Theorem 2.2.1., if we substitute L=0, then we get the following.
2.2.2 Corollary: Let h:$X \to X$ be a function on a complete S-metric space (X, S). Suppose $\psi \in \Psi$ and $\phi \in \Phi$ so that

$$\begin{aligned}
S(h\xi, h\vartheta, hw) \leq \psi(max\{&S(\xi,\vartheta,w), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), \frac{1}{2}[S(\xi,\xi,h\vartheta) \\
&+ S(\vartheta,\vartheta,h\xi)]\}) - \phi(max\{S(\xi,\vartheta,w), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), \\
&\frac{1}{2}[S(\xi,\xi,h\vartheta) + S(\vartheta,\vartheta,h\xi)]\}),
\end{aligned}$$

for all $\xi, \vartheta, w \in X$. Then h has one and only one fixed point τ in X.

In the above Corollary (2.2.2.), if ψ is the identity map then we get the following.
2.2.3 Corollary: Let h:$X \to X$ be a function on a complete S-metric space (X, S). Suppose there exists a $\phi \in \Phi$ so that

$$\begin{aligned}
S(h\xi, h\vartheta, hw) \leq\ & max\{S(\xi,\vartheta,w), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), \frac{1}{2}[S(\xi,\xi,h\vartheta) + S(\vartheta,\vartheta,h\xi)]\} \\
&- \phi(max\{S(\xi,\vartheta,w), S(\xi,\xi,h\xi), S(\vartheta,\vartheta,h\vartheta), \frac{1}{2}[S(\xi,\xi,h\vartheta) + S(\vartheta,\vartheta,h\xi)]\})
\end{aligned}$$

for all $\xi, \vartheta, w \in X$. Then h has one and only one fixed point τ in X.

The following illustration helps to support Theorem 2.2.1.

2.2.4 Example: Consider X = $[0, \frac{7}{6}]$. Define S:$X^3 \to [0,\infty)$ by $S(\xi,\vartheta,w) = max\{|\xi - w|, |\vartheta - w|\}$, for all $\xi,\vartheta,w \in$ X.

Then clearly S is an S-metric on X.

Now, we define h:X $\to$ X by

$$h\xi = \begin{cases} \frac{1}{2} & \text{when } \xi \in [0,1] \\ \frac{4}{3} - \xi & \text{when } \xi \in (1, \frac{7}{6}] \end{cases}.$$

$\psi, \phi : [0,\infty) \to [0,\infty)$ are defined by

$\psi(v) = v$, $\forall v \geq 0$ and $\phi(v) = \frac{v}{1+v}$ $\forall v \geq 0$.

Now we verify that h holds inequality (2.1.1.).

Case(i) Let $\xi,\vartheta,w \in [0, 1]$.

We suppose that $\xi > \vartheta > w$, w.l.o.g.,

$S(h\xi, h\vartheta, hw) = S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$. Hence the inequality (2.1.1.) holds trivially.

Case(ii) Let $\xi,\vartheta,w \in (1, \frac{7}{6}]$.

We suppose that $\xi > \vartheta > w$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{4}{3} - w) = max\{|\frac{4}{3} - \xi - (\frac{4}{3} - w)|, |\frac{4}{3} - \vartheta - (\frac{4}{3} - w)|\} \\
&= max\{|w - \xi|, |w - \vartheta|\} = \xi - w \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&\leq S(\xi,\xi,h\xi) - \frac{S(\xi,\xi,h\xi)}{1 + S(\xi,\xi,h\xi)} = \frac{(S(\xi,\xi,h\xi))^2}{1 + S(\xi,\xi,h\xi)} \\
&\leq \frac{(M(\xi,\vartheta,w))^2}{1 + M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1 + M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)) \\
&\leq \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)) + L.\theta(\xi,\vartheta,w).
\end{aligned}$$

Case(iii) Let $\vartheta, w \in [0, 1]$ and $\xi \in (1, \frac{7}{6}]$.

We suppose that $\vartheta > w$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{1}{2}, \frac{1}{2}) = max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{1}{2} - \frac{1}{2}|\} \\
&= \xi - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&\leq S(\xi,\xi,h\xi) - \frac{S(\xi,\xi,h\xi)}{1 + S(\xi,\xi,h\xi)} = \frac{(S(\xi,\xi,h\xi))^2}{1 + S(\xi,\xi,h\xi)} \\
&\leq \frac{(M(\xi,\vartheta,w))^2}{1 + M(\xi,\vartheta,w)} = M(\xi,\vartheta,w) - \frac{M(\xi,\vartheta,w)}{1 + M(\xi,\vartheta,w)} \\
&= \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)) \\
&\leq \psi(M(\xi,\vartheta,w)) - \phi(M(\xi,\vartheta,w)) + L.\theta(\xi,\vartheta,w).
\end{aligned}$$

Case(iv) Let $w \in [0,\ 1]$ and $\xi, \vartheta \in (1,\ \frac{7}{6}]$.

We suppose that $\vartheta > \xi$, w.l.o.g.,

$$S(h\xi, h\vartheta, hw) = S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{1}{2}) = max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{4}{3} - \vartheta - \frac{1}{2}|\}$$

$$\begin{aligned}
&= max\{|\frac{5}{6} - \xi|, |\frac{5}{6} - \vartheta|\} = \xi - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&\leq S(\vartheta, , \vartheta, h\vartheta) - \frac{S(\vartheta, , \vartheta, h\vartheta)}{1 + S(\vartheta, , \vartheta, h\vartheta)} = \frac{(S(\vartheta, , \vartheta, h\vartheta))^2}{1 + S(\vartheta, , \vartheta, h\vartheta)} \\
&\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case(v) Let $\xi, \vartheta \in [0,\ 1]$ and $w \in (1,\ \frac{7}{6}]$.

We suppose that $\xi > \vartheta$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= (\frac{1}{2}, \frac{1}{2}, \frac{4}{3} - w) = max\{|\frac{1}{2} - (\frac{4}{3} - w)|, |\frac{1}{2} - (\frac{4}{3} - w)|\} \\
&= w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\
&\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case(vi) Let $\xi \in [0,\ 1]$ and $\vartheta, w \in (1,\ \frac{7}{6}]$.

We suppose that $w > \vartheta$, w.l.o.g.,

$$\begin{aligned}
S(h\xi, h\vartheta, hw) &= S(\frac{1}{2}, \frac{4}{3} - \vartheta, \frac{4}{3} - w) = max\{|\frac{1}{2} - (\frac{4}{3} - w)|, |\frac{4}{3} - \vartheta - (\frac{4}{3} - w)|\} \\
&= max\{w - \frac{5}{6}, |w - \vartheta|\} = w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} = \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
&\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

Case(vii) Let $\vartheta \in [0,\ 1]$ and $\xi, w \in (1,\ \frac{7}{6}]$.

We suppose that $w > \xi$, w.l.o.g.,

$$S(h\xi, h\vartheta, hw) = S(\frac{4}{3} - \xi, \frac{1}{2}, \frac{4}{3} - w) = max\{|\frac{4}{3} - \xi - (\frac{4}{3} - w)|, |\frac{1}{2} - (\frac{4}{3} - w)|\}$$

$$\begin{aligned}
&= max\{|w - \xi|, w - \frac{5}{6}\} = w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
&= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} = \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
&\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
&= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) \\
&\leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L.\theta(\xi, \vartheta, w).
\end{aligned}$$

We infer from all of the aforementioned cases that h will be a (ψ, ϕ) - almost weakly generalized contractive map on X and that $\frac{1}{2}$ is its only fixed point.

Chapter 3

Fixed point results for Z_S-contractions in relation to simulation function in S-metric spaces

3.1 Introduction:

We prove a fixed point theorem in this chapter by defining generalized Z_s -contractions in relation to the simulation function in an S-metric space. Satish Shukla, F.Khajasteh and S.Radenovic [81] proposed the simulation function and the notion of Z-contraction in relation to simulation function in 2015 and derived a fixed point theorem that generalizes the Banach contraction principle. Very recently, Murat Olgun, O.Bicer and T.Alyildiz [82] defined generalized Z-contraction in relation to the simulation function and established a fixed point theorem. In the year 2019, Nihal Tas, Nihal Ylimaz Ozgur and Nabil Mlaiki [83] proved a fixed point theorem by employing the collection of simulation mappings on S-metric spaces.

We generalized the findings of Nihal Yilmaz Ozgur, Nihal Tas, and N.Mlaiki [83] in this work. In addition, we provide an example that validates our findings.

3.1.1 Definition: [81] Consider the function $\gamma : [0,\infty) \times [0,\infty) \to \mathbb{R}$. Then γ is said to be a simulation function if
(γ1) $\gamma(0,0) = 0$
(γ2) $\gamma(a,b) < b - a$, for a,b > 0
(γ3) If the sequences $\{a_n\}, \{b_n\}$ of (0,∞) so that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$, then $\lim_{n\to\infty}$sup $\gamma(a_n, b_n) < 0$.

We indicate Z as the collection of all simulation mappings. For example, $\gamma(a,b) = \tau b - a$ for $0 \leq \tau < 1$ belonging to Z.

Nihal Tas, N.Y.Ozgur and Nabil Mlaiki [83] defined the Z_s-contraction and proved the following theorem .
3.1.2 Definition: [83] Consider a function h:$X \to X$ on an S-metric space (X, S) and $\gamma \in$ Z. Then we say h is a Z_s-contraction in relation to γ if

$$\gamma(S(h\xi, h\xi, h\vartheta), S(\xi, \xi, \vartheta)) \geq 0 \text{ for all } \xi, \vartheta \in \text{X}.$$

3.1.3 Theorem: [83] Let h represent a self map on an S-metric space (X, S). Then h has one and only one fixed point a$\in X$ and the fixed point is the limit of the Picard sequence $\{\xi_n\}$, whenever h is a Z_s-contraction in relation to γ.

Now, we define generalized Z_s-contraction in relation to γ as follows:

3.1.4 Definition: Consider h:$X \to X$, a function on an S-metric space (X, S) and $\gamma \in$ Z. Then we say h is a generalized Z_s-contraction in relation to γ if

$$\gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) \geq 0 \quad for\ all \quad \xi, \vartheta \in X \qquad (3.1.1)$$

where $M(\xi, \xi, \vartheta) = \max\{S(\xi, \xi, \vartheta), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}$

3.1.5 Example: Let h be a contraction on (X, S). If we take L$\in$[0, 1) and γ(a, b) = L.b - a for all $0 \leq a, b < \infty$, then h is a Z_s-contraction in relation to γ. In fact, consider a = $S(h\xi, h\xi, h\vartheta)$ and b = $M(\xi, \xi, \vartheta)$. Since h is a contraction, we obtain :

$$\begin{aligned} S(h\xi, h\xi, h\vartheta) \leq LS(\xi, \xi, \vartheta) &\leq LM(\xi, \xi, \vartheta) \\ \implies LM(\xi, \xi, \vartheta) &- S(h\xi, h\xi, h\vartheta) \geq 0 \\ \implies \gamma(S(h\xi, h\xi, h\vartheta)&, M(\xi, \xi, \vartheta)) \geq 0. \end{aligned}$$

for all $\xi, \vartheta \in X$. Therefore, h is a generalized Z_s-contraction in relation to γ.

3.1.6 Example: Consider the complete S-metric space (X, S), where
X = [0, 1] and $S : X^3 \to [0, \infty)$ by $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$.
Define h:$X \to X$ by

$$h\xi = \begin{cases} \frac{2}{5}, \text{for } \xi \in [0, \frac{2}{3}) \\ \frac{1}{5}, \text{for } \xi \in [\frac{2}{3}, 1) \end{cases}$$

Now we prove that h is a generalized Z_s-contraction in relation to γ, where γ is defined by $\gamma(a, b) = \frac{6}{7}b - a$. Now

$$\begin{aligned} S(h\xi, h\xi, h\vartheta) &\leq \frac{3}{7}[S(\xi, \xi, h\xi) + S(\vartheta, \vartheta, h\vartheta)] \\ &\leq \frac{6}{7} max\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta)\} \\ &\leq \frac{6}{7} M(\xi, \xi, \vartheta) \end{aligned}$$

for all $\xi, \vartheta \in X$.

That is, we have

$$\gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)) = \frac{6}{7} M(\xi, \xi, \vartheta) - d(h\xi, h\xi, h\vartheta) \geq 0.$$

for all $\xi, \vartheta \in X$.

3.1.7 Definition: Let (X, S) be an S-metric space. Then we say that a mapping h:$X \to X$ is asymptotically regular at $\xi \in$X if $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$.

3.2 Main Results and Examples

We derive two Lemmas and a fixed point theorem using generalized Z_s-contraction in relation to simulation function, in this section. By the following Lemma 3.2.1, we can see that a generalized Z_s-contraction is asymptotically regular at every point of X. In Lemma 3.2.2, we prove that if h is a generalized Z_s-contraction in relation to simulation function then the Picard's sequence generated by h is a bounded sequence.

3.2.1 Lemma: If $h : X \to X$ is a generalized Z_s-contraction in relation to γ, then h is an asymptotically regular at each point $\xi \in X$.

Proof: Let $\xi \in X$. If for some m $\in$N, we have $h^m\xi = h^{m-1}\xi$, that is, h$\vartheta = \vartheta$, where $\vartheta = h^{m-1}\xi$, then

$h^n\vartheta = h^{n-1}h\vartheta = h^{n-1}\vartheta = ... =$ h$\vartheta = \vartheta$ for each n$\in$N. Therefore, we have:

$$\begin{aligned} S(h^n\xi, h^n\xi, h^{n+1}\xi) &= S(h^{n-m+1}h^{m-1}\xi, h^{n-m+1}h^{m-1}\xi, h^{n-m+2}h^{m-1}\xi) \\ &= S(h^{n-m+1}\vartheta, h^{n-m+1}\vartheta, h^{n-m+2}\vartheta) \\ &= S(\vartheta, \vartheta, \vartheta) \\ &= 0 \end{aligned}$$

Hence

$$\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$$

Now, we assume that $h^n\xi \neq h^{n+1}\xi$, for each n$\in$N.

From the condition(γ2) and the generalized Z_s-contraction property, we get:

$$0 \leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \qquad (3.2.1)$$

where

$$\begin{aligned} M(h^n\xi, h^n\xi, h^{n-1}\xi) &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, hh^n\xi), S(h^{n-1}\xi, h^{n-1}\xi, hh^{n-1}\xi), \\ &\quad \frac{1}{2}[S(h^n\xi, h^n\xi, hh^{n-1}\xi) + S(h^{n-1}\xi, h^{n-1}\xi, hh^n\xi)]\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^n\xi, h^n\xi, h^{n+1}\xi), S(h^{n-1}\xi, h^{n-1}\xi, h^n\xi), \\ &\quad \frac{1}{2}[S(h^n\xi, h^n\xi, h^n\xi) + S(h^{n-1}\xi, h^{n-1}\xi, h^{n+1}\xi)\} \\ &= max\{S(h^n\xi, h^n\xi, h^{n-1}\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)\} \end{aligned}$$

If $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) > S(h^n\xi, h^n\xi, h^{n-1}\xi)$, then we get

$M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi)$

From equation (3.2.1) we have,

$$0 \leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi))$$
$$< S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) = 0$$

which is a contradiction.
Hence, $M(h^n\xi, h^n\xi, h^{n-1}\xi) = S(h^n\xi, h^n\xi, h^{n-1}\xi)$.
Using generalized Z_s-contractive property, we get

$$\begin{aligned} 0 &\leq \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &= \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi)) \\ &< S(h^n\xi, h^n\xi, h^{n-1}\xi) - S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) \end{aligned}$$

i.e., $S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi) < S(h^n\xi, h^n\xi, h^{n-1}\xi)$ for all n∈N.
Then $\{S(h^n\xi, h^n\xi, h^{n-1}\xi)\}$ is a non-negative reals decreasing sequence and so it will be convergent. Suppose $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = \eta \geq 0$. If $\eta > 0$, then from the condition ($\gamma 3$) and the generalized Z_s-contraction property, we get

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), M(h^n\xi, h^n\xi, h^{n-1}\xi) \\ &= \lim_{n\to\infty} sup\ \gamma(S(h^{n+1}\xi, h^{n+1}\xi, h^n\xi), S(h^n\xi, h^n\xi, h^{n-1}\xi) < 0 \end{aligned}$$

which is a contradiction. So $\eta = 0$.
Therefore $\lim_{n\to\infty} S(h^n\xi, h^n\xi, h^{n+1}\xi) = 0$.
Hence, h is asymptotically regular at each point $\xi \in X$.

3.2.2 Lemma: The Picard sequence $\{\xi_n\}$, where h$\xi_{n-1} = \xi_n$, to each n∈N and the initial point $\xi_0 \in X$, is a bounded sequence, whenever h is a generalized Z_s-contraction in relation to γ.

Proof: Consider the Picard sequence $\{\xi_n\}$ in X with initial value ξ_0. Now we claim that $\{\xi_n\}$ is a bounded sequence.
Assume that $\{\xi_n\}$ is unbounded. Let $\xi_{n+m} \neq \xi_n$, for each m,n∈N.
Since $\{\xi_n\}$ is unbounded, we can find a sub sequence $\{\xi_{n_l}\}$ of $\{\xi_n\}$ so that $n_1 = 1$ and to each l ∈N, n_{l+1} is the smallest integer so that
$S(\xi_{n_l+1}, \xi_{n_l+1}, \xi_{n_l}) > 1$ and $S(\xi_m, \xi_m, \xi_{n_l}) \leq 1$ for $n_l \leq m \leq n_{l+1} - 1$
Hence, from the Lemma (1.2.17.), we obtain

$$\begin{aligned} 1 &< S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_l}) \\ &\leq 2S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_{l+1}-1}) + S(\xi_{n_l}, \xi_{n_l}, \xi_{n_{l+1}-1}) \\ &\leq 2S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_{l+1}-1}) + 1 \end{aligned}$$

Letting $l \to \infty$ and using Lemma (3.2.1), we have

$$\lim_{n\to\infty} S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_l}) = 1.$$

Now

$$\begin{aligned}
&1 < S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_l}) \le M(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1}) \\
&= max\{S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1}), S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_{l+1}}), S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}), \\
&\frac{1}{2}[S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l}) + S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_{l+1}})]\} \\
&= max\{S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_{l+1}-1}), S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_{l+1}}), S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}), \\
&\frac{1}{2}[S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l}) + S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_{l+1}})]\} \\
&\le max\{2S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}) + S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l}), S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_{l+1}}), \\
&S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}), \frac{1}{2}[S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l}) + S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_{l+1}})]\} \\
&\le max\{2S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}) + 1, S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_{l+1}}), \\
&S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}), \frac{1}{2}[1 + 2S(\xi_{n_l-1}, \xi_{n_l-1}, \xi_{n_l}) + S(\xi_{n_l}, \xi_{n_l}, \xi_{n_{l+1}})]\}
\end{aligned}$$

Letting n$\to \infty$, we get

$$1 \le \lim_{l\to\infty} M(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1}) \le 1.$$

That is, $\lim_{l\to\infty} M(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1}) = 1$.
From the condition($\gamma 3$) and the generalized Z_s-contraction property, we obtain

$$\begin{aligned}
0 &\le \lim_{l\to\infty} sup\ \gamma(S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_l}), M(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1})) \\
&= \lim_{l\to\infty} sup\ \gamma(S(\xi_{n_{l+1}}, \xi_{n_{l+1}}, \xi_{n_l}), S(\xi_{n_{l+1}-1}, \xi_{n_{l+1}-1}, \xi_{n_l-1})) < 0
\end{aligned}$$

This is a contradiction. Hence our assumption is wrong.
Therefore $\{\xi_n\}$ is bounded.

3.2.3 Theorem: Suppose that h:$X \to X$ is a mapping defined on a complete S-metric space (X, S). Then h has one and only one fixed point $\eta \in X$ and Picard sequence $\{\xi_n\}$ converges to the fixed point η, whenever h is a generalized Z_s-contraction in relation to γ.

Proof. Let the Picard sequence $\{\xi_n\}$ be defined as h$\xi_{n-1} = \xi_n$, $\forall n \in$N and $\xi_0 \in X$. Now, we verify that $\{\xi_n\}$ is a Cauchy sequence. To get this, consider

$$T_n = sup\{S(\xi_i, \xi_i, \xi_j) : i, j \ge n\}.$$

Clearly $\{T_n\}$ is a non negative reals decreasing sequence. Hence, we can find a $\tau \geq 0$ so that $\lim_{n\to\infty} T_n = \tau$. Now we prove that $\tau = 0$. If possible, suppose that $\tau > 0$. From the definition of T_n, to each k∈N, we can find m_k, n_k so that $k \leq n_k < m_k$ and

$$T_k - \frac{1}{k} < S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \leq T_k$$

Therefore, we get $\lim_{n\to\infty} S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) = \tau$.

From the Lemma (1.2.17.), Lemma (3.2.1) and generalized Z_s-contraction property, we get

$$\begin{aligned}
S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) &\leq S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \\
&\leq 2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{m_k}) \\
&\leq 2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + 2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k})
\end{aligned}$$

Letting as k→ ∞, we have

$$\lim_{k\to\infty} S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \tau.$$

Now

$$\begin{aligned}
&S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \leq M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) \\
&= max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, h\xi_{m_k-1}), S(\xi_{n_k-1}, \xi_{n_k-1}, h\xi_{n_k-1}), \\
&\frac{1}{2}[S(\xi_{m_k-1}, \xi_{m_k-1}, h\xi_{n_k-1}) + S(\xi_{n_k-1}, \xi_{n_k-1}, h\xi_{m_k-1})]\} \\
&= max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\
&\frac{1}{2}[S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) + S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{m_k})]\} \\
&\leq max\{S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}), S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}), S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}), \\
&\frac{1}{2}[2S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{m_k}) + S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) + \\
&2S(\xi_{n_k-1}, \xi_{n_k-1}, \xi_{n_k}) + S(\xi_{n_k}, \xi_{n_k}, \xi_{m_k})]\}
\end{aligned}$$

Letting k → ∞, we get

$$\lim_{k\to\infty} M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \tau.$$

From the condition ($\gamma 3$) and the generalized Z_s-contraction property, we have

$$0 \leq \lim_{k\to\infty} sup\ \gamma(S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}), M(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1})) < 0$$

This is a contraction, Hence, $\tau = 0$.

This claims that the sequence $\{\xi_n\}$ becomes a Cauchy in X and since X is complete, we can find $\eta \in X$ so that $\lim_{n\to\infty} \xi_n = \eta$.

Now we verify that, η is a fixed point of h.

If suppose h$\eta \neq \eta$, then $S(\eta, \eta, h\eta) = S(h\eta, h\eta, \eta) > 0$.

Now,

$$M(\xi_n, \xi_n, \eta) = max\{S(\xi_n, \xi_n, \eta), S(\xi_n, \xi_n, h\xi_n), S(\eta, \eta, h\eta), \frac{1}{2}[S(\xi_n, \xi_n, h\eta) + S(\eta, \eta, h\xi_n)]\}$$

$$\begin{aligned}\lim_{n\to\infty} M(\xi_n, \xi_n, \eta) &= max\{S(\eta, \eta, \eta), S(\eta, \eta, \eta), S(\eta, \eta, h\eta), \frac{1}{2}[S(\eta, \eta, h\eta) + S(\eta, \eta, \eta)]\} \\ &= S(\eta, \eta, h\eta)\end{aligned}$$

From the conditions $(\gamma 2), (\gamma 3)$ and Z_s-contraction property, we get

$$0 \leq \lim_{n\to\infty} sup\ \gamma(S(h\xi_n, h\xi_n, h\eta), M(\xi_n, \xi_n, \eta)) < 0$$

This is contradiction. Hence $S(\eta, \eta, h\eta) = 0 \implies$ h$\eta = \eta$.

Hence, η is a fixed point of h.

Now we show that η is the one and only one fixed point. Suppose that τ in X so that $\tau \neq \eta$ and h$\tau = \tau$.

Now,

$$\begin{aligned}M(\eta, \eta, \tau) &= max\{S(\eta, \eta, \tau), S(\eta, \eta, h\eta), S(\tau, \tau, h\tau), \frac{1}{2}[S(\eta, \eta, h\tau) + S(\tau, \tau, h\eta]\} \\ &= max\{S(\eta, \eta, \tau), S(\eta, \eta, \eta), S(\tau, \tau, \tau), \frac{1}{2}[S(\eta, \eta, \tau) + S(\tau, \tau, \eta)]\} \\ &= S(\eta, \eta, \tau)\end{aligned}$$

From the condition $(\gamma 2)$ and Z_s-contraction property, we get

$$\begin{aligned}0 &\leq \gamma(S(h\eta, h\eta, h\tau), M(\eta, \eta, \tau)) = \gamma(S(h\eta, h\eta, h\tau), S(\eta, \eta, \tau)) \\ &< S(\eta, \eta, \tau) - S(\eta, \eta, \tau) = 0,\end{aligned}$$

This is a contradiction. It should be $\eta = \tau$.

3.2.4 Example: Consider a complete S-metric space (X, S), where X = $[0, \frac{1}{4}]$ and $S : X^3 \to [0, \infty)$ by $S(\xi, \vartheta, w) = |\xi - w| + |\xi - 2\vartheta + w|$. Define h: $X \to X$ by $h\xi = \frac{\xi}{1+\xi}$. From example 2.9 in [84], h is a Z-contraction in relation to $\gamma \in Z$,

where $\gamma(a,b) = \frac{b}{b+\frac{1}{4}} - a$, for any a,b $\in [0,\infty)$.
Therefore for all $\xi, \vartheta \in X$, we get

$$\begin{aligned} 0 &\leq \gamma(S(h\xi, h\xi, h\vartheta), S(\xi, \xi, \vartheta)) \\ &= \frac{S(\xi, \xi, \vartheta)}{S(\xi, \xi, \vartheta) + \frac{1}{4}} - S(h\xi, h\xi, h\vartheta) \\ &\leq \frac{M(\xi, \xi, \vartheta)}{M(\xi, \xi, \vartheta) + \frac{1}{4}} - S(h\xi, h\xi, h\vartheta) \\ &= \gamma(S(h\xi, h\xi, h\vartheta), M(\xi, \xi, \vartheta)). \end{aligned}$$

Thus, h is a generalized Z_s-contraction in relation to γ, for some $\gamma \in Z$. So, by the Theorem (3.2.3), h has one and only one fixed point η=0.

Chapter 4

Fixed point results for (ψ, ϕ)-weakly contractive generalized maps in S_b-metric spaces

4.1 Introduction:

We introduce (ψ, ϕ) - weakly generalized contraction map in an S_b-metric space and derived a fixed point theorem for such maps in this chapter. In 2008, Dutta et al. [85] defined (ψ, ϕ) - weekly contractive maps and obtained some fixed point results for such contractions. Later, in the year 2017, B.K.Leta and G.V.R.Babu [87] defined the following (α, ψ, ϕ) - weakly generalized contractive maps on S-metric spaces and proved a fixed point theorem for such maps as follows.

In this chapter, we indicate:
(i)$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi$ is non decreasing, continuous and ψ(t)=0 $\iff$ t=0.$\}$
(ii) $\Phi = \{\phi : [0, \infty) \to [0, \infty)$: ϕ is continuous, $\phi(t) = 0 \iff t = 0\}$.

4.1.1 Definition: [87] Consider h:$X \to X$, a function on an S-metric space (X, S). Suppose that $\exists\ \phi \in \Phi$, $\psi \in \Psi$ and $\alpha \in (0, 1)$ so that
$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(P_\alpha(\xi, \vartheta, w)) - \phi(P_\alpha(\xi, \vartheta, w))$
where $P_\alpha(\xi, \vartheta, w) = \max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw),$
$\alpha S(h\xi, h\xi, \vartheta) + (1 - \alpha)S(h\vartheta, h\vartheta, w)\}$, $\forall \xi, \vartheta, w \in X$.
Then h is called an (α, ψ, ϕ) - weakly generalized contraction map on X.

4.1.2 Theorem: [87] Consider h:$X \to X$, a function on an S-metric space (X, S). If h satisfies (α, ψ, ϕ)- weakly generalized contraction map, then h has one and only one fixed point in X.

By the motivation of B.K.Leta and G.V.R.Babu [87] results in S-metric spaces, we establish the (ψ, ϕ) - weakly generalized contraction map in S_b-metric spaces and give an example which satisfy the contraction.

4.1.3 Definition: Let (X, S_b) be an S_b-metric space for s$\geq$1. Let h be a self map of X. Then we say h is a (ψ, ϕ)- weakly generalized contraction map if $\exists$ L$\geq$0, $\psi \in \Psi$ and $\phi \in \Phi$ so that
$$\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w) \quad (4.1.1.)$$
where P$(\xi, \vartheta, w) = \max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$

$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw)+S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}$
and $Q(\xi,\vartheta,w)=\min\{S_b(hw,\xi,\xi),S_b(h\xi,\vartheta,\vartheta),S_b(h\xi,w,w),S_b(h\xi,\vartheta,w)\}$
$\forall\xi,\vartheta,w\in X$.

4.1.4 Example: Consider (X, S_b), a complete S_b-metric space for s=4, where $X=[0,\ \frac{7}{3}]$ and $S_b:X^3\to$ R is defined by
$S_b(\xi,\vartheta,w)=\frac{1}{16}[|\xi-\vartheta|+|\vartheta-w|+|w-\xi|]^2$, $\forall\xi,\vartheta,w\in X$.
We define a self map h on X by

$$h\xi=\begin{cases}\frac{1}{8} & \text{when } \xi\in[0,\ 2]\\ \frac{\xi}{16}-\frac{1}{32} & \text{when } \xi\in(2,\ \frac{7}{3}]\end{cases}.$$

Also, consider two mappings $\phi,\psi:[0,\infty)\to[0,\infty)$ defined by $\psi(\upsilon)=\upsilon$ and $\phi(\upsilon)=\frac{\upsilon}{4}\ \forall\ \upsilon\in[0,\infty)$.
Now, we check the inequality (4.1.1.)
Case(i) when $\xi,\vartheta,w\in[0,\ 2]$, we have $\psi(4s^4S_b(h\xi,h\vartheta,hw))=0$.
Then inequality (4.1.1.) holds good.
Case(ii) $\xi,\vartheta,w\in(2,\ \frac{7}{3}]$. Suppose that $\xi>\vartheta>w$. Then

$$\begin{aligned}\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16}-\frac{\vartheta}{16}|+|\frac{\vartheta}{16}-\frac{w}{16}|+|\frac{w}{16}-\frac{\xi}{16}|]^2\\ &\le \frac{4^5}{16}[3|\frac{\xi}{16}-\frac{w}{16}|]^2\\ &\le \frac{9}{4}|\xi-w|^2=\frac{1}{4}\\ &\le \frac{15123}{16384}=\frac{5041}{4096}-\frac{5041}{16384}\\ &= \frac{3}{4}S_b(\xi,\xi,h\xi)\le\frac{3}{4}P(\xi,\vartheta,w)\\ &= P(\xi,\vartheta,w)-\frac{1}{4}P(\xi,\vartheta,w)\\ &= \psi(P(\xi,\vartheta,w))-\phi(P(\xi,\vartheta,w))\\ &\le \psi(P(\xi,\vartheta,w))-\phi(P(\xi,\vartheta,w))+L.Q(\xi,\vartheta,w).\end{aligned}$$

Case(iii) $\xi,\vartheta\in[0,\ 2]$ and $w\in(2,\ \frac{7}{3}]$. Suppose that $\xi>\vartheta$. Then

$$\begin{aligned}\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5S_b(\frac{1}{8},\frac{1}{8},\frac{w}{16}-\frac{1}{32})\\ &= \frac{4^5}{16}[|\frac{1}{8}-(\frac{w}{16}-\frac{1}{32})|+|\frac{w}{16}-\frac{1}{32}-\frac{1}{8}|]\\ &= \frac{4^5}{16}[2|\frac{1}{8}-\frac{w}{16}+\frac{1}{32}|]^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}[5-2w]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w,w,hw) - \frac{1}{4}S_b(w,w,hw) \\
&= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w) \\
&= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) \\
&\leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w).
\end{aligned}
$$

Case(iv) $\vartheta, w \in [0,\ 2]$ and $\xi \in (2,\ \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$
\begin{aligned}
\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5S_b(\frac{\xi}{16}-\frac{1}{32},\frac{1}{8},\frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{\xi}{16}-\frac{1}{32}-\frac{1}{8}|+|0|+|\frac{1}{8}-(\frac{\xi}{16}-\frac{1}{32})|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8}-(\frac{\xi}{16}-\frac{1}{32})|]^2 \\
&= 4^4[\frac{5-2\xi}{32}]^2 = \frac{1}{4}[5-2\xi]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= \frac{3}{4}S_b(\xi,\xi,h\xi) \leq \frac{3}{4}P(\xi,\vartheta,w) \\
&= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w) \\
&= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) \\
&\leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w).
\end{aligned}
$$

Case(v) w$\in$[0, 2] and $\xi, \vartheta \in (2,\ \frac{7}{3}]$. Suppose that $\xi > \vartheta$. Then

$$
\begin{aligned}
\psi(4s^4S_b(h\xi,h\vartheta,hw)) &= 4^5S_b(\frac{\xi}{16}-\frac{1}{32},\frac{\vartheta}{16}-\frac{1}{32},\frac{1}{8}) \\
&= \frac{4^5}{16}[\frac{\xi-\vartheta}{16}+\frac{2\vartheta-5}{32}+\frac{5-2\xi}{32}]^2 \\
&= \frac{4^5}{16}[\frac{10-4\vartheta}{32}]^2 \\
&\leq \frac{1}{4}[5-2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w,w,hw) - \frac{1}{4}S_b(w,w,hw) \\
&= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w) \\
&= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) \\
&\leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w).
\end{aligned}
$$

Case(vi) $\xi \in [0,\ 2]$ and $\vartheta, w \in (2,\ \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[\frac{5-2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5-2w}{32}]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(vii) $\xi, w \in [0,\ 2]$ and $\vartheta \in (2,\ \frac{7}{3}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2\vartheta)]^2 \\
&= \frac{1}{4}[5-2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384} \\
&= S_b(w, w, hw) - \frac{1}{4} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{4} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(viii) $\xi \in [0,\ 2]$ and $\vartheta, w \in (2,\ \frac{7}{3}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}.\frac{1}{32^2}[2(5-2w)]^2
\end{aligned}$$

$$= \frac{1}{4}[5-2w]^2$$
$$\leq \frac{1}{4} \leq \frac{15123}{16384} = \frac{5041}{4096} - \frac{5041}{16384}$$
$$= S_b(w,w,hw) - \frac{1}{4}S_b(w,w,hw)$$
$$= P(\xi,\vartheta,w) - \frac{1}{4}P(\xi,\vartheta,w)$$
$$= \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w))$$
$$\leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w)) + L.Q(\xi,\vartheta,w).$$

Therefore h is a (ψ, ϕ) - weakly generalized contractive map.

4.2 Main Results and Examples

We establish a fixed point theorem using the (ψ, ϕ) - weakly generalized contraction maps in this section. Further, we derive some corollaries and give an example to support the result.

4.2.1 Theorem: Suppose h is a self map in a complete symmetric S_b-metric space (X, S_b) for s≥1. If h is a (ψ, ϕ) - weakly generalized contraction map, then h has one and only one fixed point in X.

Proof: Consider $\xi_0 \in X$ and a sequence $\{\xi_\ell\}$ in X is defined by $\xi_\ell = h\xi_{\ell-1}$, for ℓ =1,2,3...

Suppose $\xi_{\ell-1} = \xi_\ell$ for some ℓ. Then h has a fixed point $\xi_{\ell-1}$.

Now, we suppose that $\xi_{\ell-1} \neq \xi_\ell$, $\forall\ \ell \in$ N.

By choosing $\xi = \vartheta = \xi_{\ell-2}, w = \xi_{\ell-1}$ in (4.1.1.), we obtain

$$\begin{aligned}\psi(S_b(\xi_{\ell-1}\xi_{\ell-1},\xi_\ell)) &\leq \psi(4s^4 S_b(h\xi_{\ell-2},h\xi_{\ell-2},h\xi_{\ell-1}))\\ &\leq \psi(P(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1})), -\phi(P(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}))\\ &+ L.Q(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}) \qquad (4.2.1.)\end{aligned}$$

where

$$\begin{aligned}P(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}) = max\{&S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}), S_b(\xi_{\ell-2},\xi_{\ell-2},h\xi_{\ell-2}), S_b(\xi_{\ell-2},\xi_{\ell-2},h\xi_{\ell-2}),\\ &S_b(\xi_{\ell-1},\xi_{\ell-1},h\xi_{\ell-1}), \frac{1}{4s^2}[S_b(h\xi_{\ell-2},h\xi_{\ell-2},h\xi_{\ell-1})+\\ &S_b(h\xi_{\ell-2},h\xi_{\ell-2},\xi_{\ell-2})S_b(h\xi_{\ell-2},h\xi_{\ell-2},\xi_{\ell-1})S_b(h\xi_{\ell-1},h\xi_{\ell-1}.,\xi_{\ell-2})]\})\\ &= max\{S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}), S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)\} \qquad (4.2.2.)\end{aligned}$$

and

$$Q(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}) = min\{S_b(h\xi_{\ell-1},\xi_{\ell-2},\xi_{\ell-2}), S_b(h\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-2}),$$
$$S_b(h\xi_{\ell-2},\xi_{\ell+1},\xi_{\ell-1}), S_b(h\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1})\}$$
$$= 0. \qquad (4.2.3.)$$

If $S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)$ is the maximum in (4.2.2.) and using (4.2.3.) and (4.2.1.), we get

$$\psi(S_b(\xi_{\ell-1}\xi_{\ell-1},\xi_\ell)) \leq \psi(S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)) - \phi(S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)).$$

This implies $\phi(S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)) = 0$. Therefore, $\xi_{\ell-1} = \xi_\ell$, a contradiction to our assumption. Thus,

$$\psi(S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)) \leq \psi(S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1})) - \phi(S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1})). \qquad (4.2.4.)$$
$$< \psi(S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1})).$$

By the definition of ψ, we have

$$S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell) < S_b(\xi_{\ell-2},\xi_{\ell-2},\xi_{\ell-1}).$$

Thus, $\{S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell)\}$ is a positive real numbers strictly decreasing sequence. Then we find a p ≥ 0 so that

$$\lim_{\ell\to\infty} S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell) = p.$$

Taking $\ell \to \infty$ in (4.2.4.), we get
$\psi(p) \leq \psi(p) - \phi(p)$. This implies $\phi(p) = 0$. Hence p $= 0$. Thus,

$$\lim_{\ell\to\infty} S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_\ell) = 0. \qquad (4.2.5.)$$

By choosing $\xi = \vartheta = \xi_{\ell-1}, w = \xi_{\ell-2}$ in (4.1.1.), we get

$$\psi(S_b(\xi_\ell,\xi_\ell,\xi_{\ell-1})) \leq \psi(4s^4 S_b(h\xi_{\ell-1},h\xi_{\ell-1},h\xi_{\ell-2}))$$
$$\leq \psi(P(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2})) - \phi(P(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2}))$$
$$+ L.Q(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2}) \qquad (4.2.6.)$$

where

$$P(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2}) = max\{S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2}), S_b(\xi_{\ell-1},\xi_{\ell-1},h\xi_{\ell-1}), S_b(\xi_{\ell-1},\xi_{\ell-1},h\xi_{\ell-1}),$$
$$S_b(\xi_{\ell-2},\xi_{\ell-2},h\xi_{\ell-2}), \frac{1}{4s^2}[S_b(h\xi_{\ell-1},h\xi_{\ell-1},h\xi_{\ell-2})+$$
$$S_b(h\xi_{\ell-1},h\xi_{\ell-1},\xi_{\ell-1})S_b(h\xi_{\ell-1},h\xi_{\ell-1},\xi_{\ell-2})S_b(h\xi_{\ell-2},h\xi_{\ell-2},\xi_{\ell-1})]\}$$
$$= max\{S_b(\xi_{\ell-1},\xi_{\ell-1},\xi_{\ell-2}), S_b(\xi_\ell,\xi_\ell,\xi_{\ell-1})\} \qquad (4.2.7.)$$

and

$$Q(\xi_{\ell+1}, \xi_{\ell-1}, \xi_{\ell-2}) = min\{S_b(h\xi_{\ell-2}, \xi_{\ell-1}, \xi_{\ell-1}), S_b(h\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-1}),$$
$$S_b(h\xi_{\ell-1}, \xi_{\ell-2}, \xi_{\ell-2}), S_b(h\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-2})\}$$
$$= 0. \qquad (4.2.8.)$$

If $S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})$ is maximum in (4.2.7.) and using (4.2.6.) and (4.2.8.), we get

$$\psi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})) \leq \psi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})) - \phi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})) + L.0$$

This implies $\phi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})) = 0$. Hence, $\xi_\ell = \xi_{\ell-1}$, a contradiction to our assumption. Thus

$$\psi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})) \leq \psi(S_b(\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-2})) - \phi(S_b(\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-2})) \qquad (4.2.9.)$$
$$\leq \psi(S_b(\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-2}))$$

By the definition of ψ, we obtain

$$S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}) < S_b(\xi_{\ell-1}, \xi_{\ell-1}, \xi_{\ell-2}).$$

Thus, $\{S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})\}$ is a positive real numbers strictly decreasing sequence. Hence, we can find p ≥ 0 so that

$$\lim_{\ell\to\infty} S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}) = p.$$

Taking $\ell \to \infty$ in (4.2.9.), we obtain

$$\psi(p) \leq \psi(p) - \phi(p).$$

This implies $\phi(p) = 0$. Therefore p $= 0$. Thus,

$$\lim_{\ell\to\infty} S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}) = 0.$$

Now we verify that $\{\xi_\ell\}$ is a S_b-Cauchy sequence in X.
Suppose that sequence $\{\xi_\ell\}$ is not S_b- Cauchy . Then $\exists\ \epsilon > 0$ and monotone increasing sequence of real numbers $m(\sigma)$ and $\ell(\sigma)$ with $\ell(\sigma) > m(\sigma) > \sigma$ so that

$$S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}) \geq \epsilon \text{ and } S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-2}) < \epsilon. \qquad (4.2.10.)$$

Now from (4.1.1.), (4.2.6) and (4.2.10.), we have

$$\psi(4s^4\epsilon) \leq \psi(4s^4 S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}))$$
$$= \psi(4s^4 S_b(h\xi_{m(\sigma)-2}, h\xi_{m(\sigma)-2}, h\xi_{\ell(\sigma)-2}))$$
$$\leq \psi(P(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2})) - \phi(P(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2}))$$
$$+ L.Q(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2})$$

where

$$P(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}) = max\{S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}), S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},h\xi_{m(\sigma)-2}),$$
$$S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},h\xi_{m(\sigma)-2}), S_b(\xi_{\ell(\sigma)-2},\xi_{\ell(\sigma)-2},h\xi_{\ell(\sigma)-2}),$$
$$\frac{1}{4s^2}[S_b(h\xi_{m(\sigma)-2},h\xi_{m(\sigma)-2},h\xi_{\ell(\sigma)-2})$$
$$+ S_b(h\xi_{m(\sigma)-2},h\xi_{m(\sigma)-2},\xi_{m(\sigma)-2})S_b(h\xi_{m(\sigma)-2},h\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2})$$
$$S_b(h\xi_{\ell(\sigma)-2},h\xi_{\ell(\sigma)-2},\xi_{m(\sigma)-2})]\})$$
$$= max\{S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}), S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{m(\sigma)-1}),$$
$$S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{m(\sigma)-1}), S_b(\xi_{\ell(\sigma)-2},\xi_{\ell(\sigma)-2},\xi_{\ell(\sigma)-1}),$$
$$\frac{1}{4s^2}[S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})$$
$$+ S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{m(\sigma)-2})S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-2})$$
$$S_b(\xi_{\ell(\sigma)-1},\xi_{\ell(\sigma)-1},\xi_{m(\sigma)-2})]\})$$

As $\sigma \to \infty$

$$lim_{\sigma\to\infty} A(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}) = max\{S_b(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}),$$
$$\frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})\}.$$

and

$$Q(\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2}) = min\{S_b(h\xi_{\ell(\sigma)-2},\xi_{m(\sigma)-2},\xi_{m(\sigma)-2}), S_b(h\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{m(\sigma)-2}),$$
$$S_b(h\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2},\xi_{\ell(\sigma)-2}), S_b(h\xi_{m(\sigma)-2},\xi_{m(\sigma)-2},\xi_{\ell(\sigma)-2})\}.$$
$$= 0.$$

If $\frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})$ is maximum,

$$\psi(4s^4 S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})) \leq \psi(\frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}))$$
$$- \phi(\frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}))$$

This implies

$$\psi(4s^4 S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})) < \psi(\frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}))$$

From the property of ψ, we have

$$4s^4 S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1}) < \frac{1}{4s^2}S_b(\xi_{m(\sigma)-1},\xi_{m(\sigma)-1},\xi_{\ell(\sigma)-1})$$

This gives rise to

$4s^4 < \frac{1}{4s^2} \Rightarrow 16s^6 < 1$, a contradiction as s ≥ 1.

Therefore, we have

$$\begin{aligned}\psi(4s^4 S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1})) &\leq \psi(S_b(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2})) \\ &\quad - \phi(S_b(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2})) \\ &< \psi(S_b(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{\ell(\sigma)-2}))\end{aligned}$$

Now using Lemma(1.3.3.), we have

$$\begin{aligned}4s^4 S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-1}) &\leq 2s S_b(\xi_{m(\sigma)-2}, \xi_{m(\sigma)-2}, \xi_{m(\sigma)-1}) \\ &\quad + s^2 S_b(\xi_{m(\sigma)-1}, \xi_{m(\sigma)-1}, \xi_{\ell(\sigma)-2}).\end{aligned}$$

Letting $\sigma \to \infty$, we get

$4s^4\epsilon \leq s^2\epsilon$, a contradiction as s$\geq$1.

Hence $\{\xi_\ell\}$ is a S_b-Cauchy sequence of complete space X. Then $\exists\ \tau \in X$ so that

$$\lim_{\ell\to\infty} \xi_\ell = \tau.$$

Now we show that $h\tau = \tau$. Suppose that $h\tau \neq \tau$. Then by Lemma (1.3.8.), we have

$$\frac{1}{2s} S_b(f\tau, f\tau, \tau) \leq \liminf_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell)$$

This implies

$$\begin{aligned}\frac{4s^4}{2s} S_b(f\tau, f\tau, \tau) &\leq 4s^4 \liminf_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell) \\ &\leq 4s^4 \limsup_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell)\end{aligned}$$

Thus

$$\begin{aligned}2s^3 S_b(h\tau, h\tau, \tau) &\leq 4s^4 \liminf_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell) \\ &\leq 4s^4 \limsup_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell)\end{aligned}$$

Using the property of ψ, we get

$$\begin{aligned}\psi(2s^3 S_b(h\tau, h\tau, \tau)) &\leq \psi(4s^4 \limsup_{\ell\to\infty} S_b(h\tau, h\tau, h\xi_\ell)) \\ &\leq \psi(\limsup_{\ell\to\infty} P(\tau, \tau, \xi_\ell)) - \phi(\limsup_{\ell\to\infty} P(\tau, \tau, \xi_\ell)) \\ &\quad + L(\limsup_{\ell\to\infty} Q(\tau, \tau, \xi_\ell))\end{aligned}$$

Now

$$
\begin{aligned}
P(\tau,\tau,\xi_\ell) &= max\{S_b(\tau,\tau,\xi_\ell), S_b(\tau,\tau,h\tau), S_b(\tau,\tau,h\tau), S_b(\xi_\ell,\xi_\ell,h\xi_\ell),\\
&\frac{1}{4s^2}[S_b(h\tau,h\tau,h\xi_\ell) + S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,\xi_\ell)S_b(h\xi_\ell,h\xi_\ell,\tau)]\}\\
&= max\{S_b(\tau,\tau,h\tau), \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)\}
\end{aligned}
$$

$$
\begin{aligned}
Q(\tau,\tau,\xi_\ell) &= min\{S_b(h\xi_\ell,\tau,\tau), S_b(h\tau,\tau,\tau), S_b(h\tau,\xi_\ell,\xi_\ell), S_b(h\tau,\tau,\xi_\ell)\}\\
&= 0
\end{aligned}
$$

If $\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)$ is maximum, we get

$$
\begin{aligned}
\psi(2s^3S_b(h\tau,h\tau,\tau)) &\leq \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) - \phi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau)) + L.0\\
&< \psi(\frac{1}{4s^2}S_b(h\tau,h\tau,\tau))
\end{aligned}
$$

Using the property of ψ, we get

$$
2s^3S_b(h\tau,h\tau,\tau) < \frac{1}{4s^2}S_b(h\tau,h\tau,\tau)
$$

this implies

$8s^5 < 1$, a contradiction. Therefore

$$
\begin{aligned}
&\psi(2s^3S_b(h\tau,h\tau,\tau)) \leq \psi(S_b(\tau,\tau,h\tau)) - \phi(S_b(\tau,\tau,h\tau)) + L.0\\
&i.e., \quad \psi(2s^3S_b(h\tau,h\tau,\tau)) < \psi(S_b(\tau,\tau,h\tau)). \qquad (4.2.11.)
\end{aligned}
$$

If $\tau \neq h\tau$, in (4.2.11.), we have

$$
2s^3S_b(h\tau,h\tau,\tau) < S_b(\tau,\tau,h\tau) \leq sS_b(h\tau,h\tau,\tau)
$$

which implies

$2s^2 < 1$, is a contradiction. Therefore, $h\tau = \tau$.

Now, we show that τ is unique.

Let τ and β be two different fixed points of h.

Now, consider

$$
\begin{aligned}
\psi(S_b(\tau,\tau,\beta)) &= \psi(S_b(h\tau,h\tau,h\beta))\\
&\leq \psi(4s^4S_b(h\tau,h\tau,h\beta)) \qquad (4.2.12.)\\
&\leq \psi(P(\tau,\beta,\beta)) - \phi(P(\tau,\beta,\beta)) + L.Q(\tau,\beta,\beta)
\end{aligned}
$$

where

$$P(\tau,\beta,\beta) = max\{S_b(\tau,\beta,\beta), S_b(\tau,\tau,h\tau), S_b(\beta,\beta,h\beta), S_b(\beta,\beta,h\beta),$$
$$\frac{1}{4s^4}[S_b(h\tau,h\beta,h\beta) + S_b(h\tau,h\tau,\tau)S_b(h\tau,h\tau,\beta)S_b(h\beta,h\beta,\beta)]\}$$
$$= \{S_b(\tau,\beta,\beta), \frac{1}{4s}S_b(\tau,\beta,\beta)\} = S_b(\tau,\beta,\beta) \qquad (4.2.13.)$$

and

$$Q(\tau,\beta,\beta) = min\{S_b(h\beta,\tau,\tau), S_b(h\tau,\beta,\beta), S_b(h\tau,f\tau,\beta), S_b(h\beta,f\beta,\beta)\} = 0 \qquad (4.2.14.)$$

From (4.2.12.),(4.2.13.) and (4.2.14.) we get

$$\psi(\frac{1}{4s^4}S_b(\tau,\beta,\beta)) \leq \psi(S_b(\tau,\beta,\beta)) - \phi(S_b(\tau,\beta,\beta)) + L.0$$
$$< \psi(S_b(\tau,\beta,\beta)).$$

From the property of ψ, we have $4s^4 < 1$, a contradiction.
There fore, we get $S_b(\tau,\beta,\beta) = 0$
Hence $\beta = \tau$. Hence τ is the one and only one fixed point of h.

In the Theorem (4.2.1.), if we substitute L=0, we get the following.

4.2.2 Corollary: Let h be a self map of X, where X is an S_b-metric space. Suppose $\exists\ \phi \in \Phi$ and $\psi \in \Psi$ so that
$\psi(4s^4 S_b(h\xi,h\vartheta,hw)) \leq \psi(P(\xi,\vartheta,w)) - \phi(P(\xi,\vartheta,w))$
where $P(\xi,\vartheta,w) = \max\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}$.
$\forall \xi,\vartheta,w \in X$. Then h contains one and only one fixed point in X.

If ψ is the identity function in the Corollary (4.2.2), we get a Corollary as follows.

4.2.3 Corollary: Let h be a self map of X, where X is an S_b-metric space. Suppose there exists $\phi \in \Phi$ so that
$4s^4 S_b(h\xi,h\vartheta,hw) \leq\ P(\xi,\vartheta,w) - \phi(P(\xi,\vartheta,w))$
where $P(\xi,\vartheta,w) = \max\{S_b(\xi,\vartheta,w), S_b(\xi,\xi,h\xi), S_b(\vartheta,\vartheta,h\vartheta), S_b(w,w,hw),$
$\frac{1}{4s^2}[S_b(h\xi,h\vartheta,hw) + S_b(h\xi,h\xi,\xi)S_b(h\xi,h\xi,w)S_b(hw,hw,\vartheta)]\}$.
$\forall\ \xi,\vartheta,w \in X$. Then h contains one and only one fixed point in X.

If we substitute P(ξ,ϑ,w) = P*(ξ,ϑ,w) in the Theorem (4.2.1.), we obtain the following.

4.2.4 Corollary: Let h be a self map of X, where X is an S_b-metric space. Suppose $\exists\ \phi \in \Phi$ and $\psi \in \Psi$ so that

$\psi(4s^4 S_b(h\xi, h\vartheta, hw)) \leq \psi(P^*(\xi, \vartheta, w)) - \phi(P^*(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w)$

where $P^*(\xi, \vartheta, w) = \max\{S_b(\xi, \vartheta, w), S_b(\xi, \xi, h\xi), S_b(\vartheta, \vartheta, h\vartheta), S_b(w, w, hw),$
$\frac{S_b(\xi,\xi,h\xi)S_b(\vartheta,\vartheta,h\vartheta)}{1+S_b(\xi,\xi,h\xi)+S_b(\xi,\vartheta,w)}, \frac{S_b(\xi,\xi,h\xi)S_b(w,w,hw)}{1+S_b(w,w,hw)+S_b(\xi,\vartheta,w)},$
$\frac{1}{4s^2}[S_b(h\xi, h\vartheta, hw) + S_b(h\xi, h\xi, \xi)S_b(h\xi, h\xi, w)S_b(hw, hw, \vartheta)]\}.$

and $Q(\xi, \vartheta, w) = \min\{S_b(hw, \xi, \xi), S_b(h\xi, \vartheta, \vartheta), S_b(h\xi, w, w), S_b(h\xi, \vartheta, w)\}$

$\forall \xi, \vartheta, w \in X$. Then h contains one and only one fixed point in X.

In Theorem (4.2.1.), if we put s=1, we get the following.

4.2.5 Corollary: Let h be a self map of X, where X is an S-metric space. Suppose that $\exists$ L$\geq 0, \phi \in \Phi$ and $\psi \in \Psi$ so that

$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w)$

where $P(\xi, \vartheta, w) = \max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw),$
$\frac{1}{2}[S(h\xi, h\vartheta, hw) + S(h\xi, h\xi, \xi)S(h\xi, h\xi, w)S(hw, hw, \vartheta)]\}$

and $Q(\xi, \vartheta, w) = \min\{S(hw, \xi, \xi), S(h\xi, \vartheta, \vartheta), S(h\xi, w, w), S(h\xi, \vartheta, w)\}$

$\forall \xi, \vartheta, w \in X$. Then h contains one and only one fixed point in X.

4.2.6 Example: Consider $X = [0,\ \frac{12}{5}]$ and define $S_b : X^3 \rightarrow$ R by

$S_b(\xi, \vartheta, w) = \frac{1}{16}[|\xi - \vartheta| + |\vartheta - w| + |w - \xi|]^2$, $\forall \xi, \vartheta, w \in X$.

Then (X, S_b) is a complete S_b-metric space for s=4.

We define a self map h on X by

$$h\xi = \begin{cases} \frac{1}{8} & \text{when } \xi \in [0, 2] \\ \frac{\xi}{16} - \frac{1}{32} & \text{when } \xi \in (2,\ \frac{12}{5}] \end{cases}.$$

Also, consider two maps $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$ defined by

$\psi(v) = v$ and $\phi(v) = \frac{v}{3}$, $\forall\ v \in [0, \infty)$.

Now, we validate the inequality (4.1.1.).

Case(i) When $\xi, \vartheta, w \in$ [0, 2], we have $\psi(4s^4 S_b(h\xi, h\vartheta, hw)) = 0$.

Then inequality (4.1.1.) holds good.

Case(ii) $\xi, \vartheta, w \in (2,\ \frac{12}{5}]$. Suppose that $\xi > \vartheta > w$. Then

$$\begin{aligned}\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5.\frac{1}{16}[|\frac{\xi}{16} - \frac{\vartheta}{16}| + |\frac{\vartheta}{16} - \frac{w}{16}| + |\frac{w}{16} - \frac{\xi}{16}|]^2 \\ &\leq \frac{4^5}{16}[3|\frac{\xi}{16} - \frac{w}{16}|]^2 \\ &\leq \frac{9}{4}|\xi - w|^2 = \frac{9}{25}\end{aligned}$$

$$\leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288}$$
$$= S_b(\xi, \xi, h\xi) - \frac{1}{3} S_b(\xi, \xi, h\xi)$$
$$= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w)$$
$$= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w))$$
$$\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).$$

Case(iii) $\xi, \vartheta \in$[0, 2] and w$\in$ (2, $\frac{12}{5}$]. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{1}{8}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{w}{16} - \frac{1}{32})| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - \frac{w}{16} + \frac{1}{32}|]^2 \\
&= \frac{1}{4}[5 - 2w]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(iv) $\vartheta, w \in$ [0, 2] and $\xi \in$ (2, $\frac{12}{5}$]. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{1}{8}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{\xi}{16} - \frac{1}{32} - \frac{1}{8}| + |0| + |\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= \frac{4^5}{16}[2|\frac{1}{8} - (\frac{\xi}{16} - \frac{1}{32})|]^2 \\
&= 4^4[\frac{5 - 2\xi}{32}]^2 = \frac{1}{4}[5 - 2\xi]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= \frac{2}{3} S_b(\xi, \xi, h\xi) \leq \frac{2}{3} P(\xi, \vartheta, w) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(v) w∈[0, 2] and $\xi, \vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > \vartheta$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{\xi}{16} - \frac{1}{32}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[\frac{\xi - \vartheta}{16} + \frac{2\vartheta - 5}{32} + \frac{5 - 2\xi}{32}]^2 \\
&= \frac{4^5}{16}[\frac{10 - 4\vartheta}{32}]^2 \\
&\leq \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(vi) ξ ∈[0, 2] and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16}[\frac{5 - 2\vartheta}{32} + \frac{2\vartheta - 2w}{32} + \frac{5 - 2w}{32}]^2 \\
&= \frac{4^5}{16} . \frac{1}{32^2}[2(5 - 2w)]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3} S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3} P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}$$

Case(vii) ξ, w ∈[0, 2] and $\vartheta \in (2, \frac{12}{5}]$. Suppose that $\xi > w$. Then

$$\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{\vartheta}{16} - \frac{1}{32}, \frac{1}{8}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta}{16} - \frac{1}{32} - \frac{1}{8}| + |0|]^2 \\
&= \frac{4^5}{16} . \frac{1}{32^2}[2(5 - 2\vartheta)]^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}[5 - 2\vartheta]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}
$$

Case(viii) $\xi \in [0, 2]$ and $\vartheta, w \in (2, \frac{12}{5}]$. Suppose that $\vartheta > w$. Then

$$
\begin{aligned}
\psi(4s^4 S_b(h\xi, h\vartheta, hw)) &= 4^5 S_b(\frac{1}{8}, \frac{y}{16} - \frac{1}{32}, \frac{w}{16} - \frac{1}{32}) \\
&= \frac{4^5}{16}[|\frac{1}{8} - (\frac{\vartheta}{16} - \frac{1}{32})| + |\frac{\vartheta - w}{16}| + |\frac{w}{16} - \frac{1}{32} - \frac{1}{8}|]^2 \\
&= \frac{4^5}{16} . \frac{1}{32^2}[2(5 - 2w)]^2 \\
&= \frac{1}{4}[5 - 2w]^2 \\
&\leq \frac{1}{4} \leq \frac{10558}{12288} = \frac{5329}{4096} - \frac{5329}{12288} \\
&= S_b(w, w, hw) - \frac{1}{3}S_b(w, w, hw) \\
&= P(\xi, \vartheta, w) - \frac{1}{3}P(\xi, \vartheta, w) \\
&= \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) \\
&\leq \psi(P(\xi, \vartheta, w)) - \phi(P(\xi, \vartheta, w)) + L.Q(\xi, \vartheta, w).
\end{aligned}
$$

Hence h holds the conditions of Theorem 4.2.1. and $\frac{1}{8}$ is the one and only one fixed point of h.

Chapter 5

Fixed and Common fixed point results in S_b-metric Spaces using implicit relation

5.1 Introduction

Sedghi and Dung [109] established a general fixed point theorem in an S-metric space using implicit relation in 2014. Later, in 2021, Gurucharan Singh Saluja [110] derived fixed point and common fixed point theorems in an S-metric space by defining the following implicit relation.

5.1.1 Definition (Implicit Relation):
Let $\Psi = \{\psi\colon \mathbb{R}_+^4 \to \mathbb{R}_+ : \psi$ is continuous and non decreasing$\}$ and for $q \in [0, 1)$. We assume the following conditions.
(1) For $\xi, \vartheta \in \mathbb{R}_+$, if $\xi \leq \psi(\vartheta, \vartheta, \xi, \frac{4\xi+\vartheta}{3})$, then $\xi \leq q\vartheta$.
(2) For $\xi \in \mathbb{R}_+$, if $\xi \leq \psi(0, \xi, 0, 0)$, then $\xi = 0$.
(3) For $\xi \in \mathbb{R}_+$, if $\xi \leq \psi(\xi, 0, 0, \frac{\xi}{3})$ then $\xi = 0$. Since $q \in [0, 1)$.

Motivated by GS Saluja [110] result, we prove fixed and common fixed point theorems in S_b-metric spaces using implicit relation in this chapter. The findings given in this research extend and generalize various findings from the previous literature.

Now, we establish an implicit relation to derive some fixed point and common fixed point theorems in S_b-metric spaces.

5.1.2 Definition(Implicit Relation):
Let $\Psi = \{\psi\colon \mathbb{R}_+^5 \to \mathbb{R}_+ : \psi$ is continuous and non decreasing$\}$ and for $q \in [0, \frac{1}{s^2}]$, where s $\geq$1. We assume the following conditions.
(R1) For $\xi, \vartheta \in \mathbb{R}_+$, if $\xi \leq \psi(\vartheta, s\xi, s\vartheta, s\xi, \xi + s\vartheta)$ then $\xi \leq q\vartheta$.
(R2) For $\xi, \vartheta \in \mathbb{R}_+$, if $\xi \leq \psi(0, 0, \xi, 0, 0)$ then $\xi = 0$.
(R3) For $\xi \in \mathbb{R}_+$, if $\xi \leq \psi(\xi, 0, 0, 0, \frac{\xi}{2})$ then $\xi = 0$.

5.2 Main Results and Examples

We prove a fixed point theorem satisfying an implicit relation in S_b-metric spaces, through this section. Further, we provide a corollary and an example to that corollary.

5.2.1 Theorem: Suppose T is a self map on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$S_b(T\xi, T\vartheta, Tw) \leq \psi(S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T\xi), S_b(w, w, Tw), S_b(\xi, \xi, T\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T\vartheta) + S_b(w, w, T\xi)]) \qquad (5.2.1)$$

for all $\xi, \vartheta, w \in$ X and $\psi \in \Psi$. Then T has one and only one fixed point in X, whenever ψ satisfies (R1), (R2), and (R3).

Proof: Consider an arbitrary $\xi_0 \in$ X and construct a sequence $\{\xi_\ell\}$ in X so that $\xi_{\ell+1} = T\xi_\ell$, to each $\ell \in \mathbb{N}$. If $\xi_{\ell+1} = \xi_\ell$, for some $\ell \in \mathbb{N}$, then $\xi_\ell = T\xi_\ell$. Hence, T has a fixed point. Now, we suppose that $\xi_{\ell+1} \neq \xi_\ell$, $\forall \ell \in \mathbb{N}$. Now utilizing the inequality (5.2.1) and Lemma 1.3.3., we consider

$$\begin{aligned} S_b(\xi_{\ell+1}., \xi_{\ell+1}, \xi_\ell) &= S_b(T\xi_\ell, T\xi_\ell, T\xi_{\ell-1}) \\ &\leq \psi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}), S_b(\xi_\ell, \xi_\ell, T\xi_\ell), S_b(\xi_{\ell-1}, \xi_{\ell-1}, T\xi_{\ell-1}), \\ &\quad S_b(\xi_\ell, \xi_\ell, T\xi_\ell), \frac{1}{2s}[S_b(\xi_\ell, \xi_\ell, T\xi_\ell) + S_b(\xi_{\ell-1}, \xi_{\ell-1}, T\xi_{\ell-1})]) \\ &= \psi(S_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}), sS_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell), sS_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}), \\ &\quad sS_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell), [S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell) + sS_b(\xi_\ell, \xi_\ell, \xi_{\ell-1})]) \end{aligned} \qquad (5.2.2)$$

Since $\psi \in \Psi$ holds the property (R1), we can find q $\in [0, \frac{1}{s^2}]$ so that

$$S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_\ell) \leq qS_b(\xi_\ell, \xi_\ell, \xi_{\ell-1}) \leq q^\ell S_b(\xi_1, \xi_1, \xi_0) \qquad (5.2.3)$$

For $\ell, m \in \mathbf{N}$ with $\ell <$ m, utilizing Lemma 1.3.3. and equation (5.2.3), we have

$$\begin{aligned} S_b(\xi_\ell, \xi_\ell, \xi_m) &\leq 2sS_b(\xi_\ell, \xi_\ell, \xi_{\ell+1}) + s^2 S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_m) \\ &\leq 2sS_b(\xi_\ell, \xi_\ell, \xi_{\ell+1}) + s^2[2S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_{\ell+2}) + s^2 S_b(\xi_{\ell+2}, \xi_{\ell+2}, \xi_m)] \\ &\leq 2sq^\ell[1 + s^2 q + (s^2 q)^2 + ...]S_b(\xi_0, \xi_0, \xi_1) \\ &\leq (\frac{2sq^\ell}{1 - s^2 q})S_b(\xi_0, \xi_0, \xi_1) \end{aligned}$$

Taking the limit as $\ell \to \infty$, we get $S_b(\xi_\ell, \xi_\ell, \xi_m) \to 0$, since q $\in [0, \frac{1}{s^2}]$ and $s \geq 1$. This claims that the sequence $\{\xi_\ell\}$ becomes a Cauchy sequence X and since X is complete, we can find a $\varrho \in X$ so that $\lim_{\ell\to\infty} \xi_\ell = \varrho$. Now we verify that ϱ is a fixed point of T. Again by utilizing inequality (5.2.1), we obtain

$$\begin{aligned}S_b(\xi_{\ell+1},\xi_{\ell+1},T\varrho) &= S_b(T\xi_\ell,T\xi_\ell,T\varrho)\\ &\leq \psi(S_b(\xi_\ell,\xi_\ell,\varrho),S_b(\xi_\ell,\xi_\ell,T\xi_\ell),S_b(\varrho,\varrho,T\varrho),\\ &S_b(\xi_\ell,\xi_\ell,T\xi_\ell),\frac{1}{2s}[S_b(\xi_\ell,\xi_\ell,T\xi_\ell)+S_b(\varrho,\varrho,T\xi_\ell)])\\ &= \psi(S_b(\xi_\ell,\xi_\ell,\varrho),S_b(\xi_\ell,\xi_\ell,\xi_{\ell+1}),S_b(\varrho,\varrho,T\varrho),\\ &S_b(\xi_\ell,\xi_\ell,\xi_{\ell+1}),\frac{1}{2s}[S_b(\xi_\ell,\xi_\ell,\xi_{\ell+1})+S_b(\varrho,\varrho,\xi_{\ell+1})])\end{aligned}$$

Letting $\ell\to\infty$, we get

$$\begin{aligned}S_b(\varrho,\varrho,T\varrho) \leq &\psi(S_b(\varrho,\varrho,\varrho),S_b(\varrho,\varrho,\varrho),S_b(\varrho,\varrho,T\varrho),\\ &S_b(\varrho,\varrho,\varrho),\frac{1}{2s}[S_b(\varrho,\varrho,\varrho)+S_b(\varrho,\varrho,\varrho)])\end{aligned}$$

$$i.e.,\ S_b(\varrho,\varrho,T\varrho)\leq\psi(0,0,S_b(\varrho,\varrho,T\varrho),0,0)$$

Since $\psi\in\Psi$ holds the property (R2), we obtain

$$S_b(\varrho,\varrho,T\varrho)\leq qS_b(\varrho,\varrho,T\varrho)$$

$$that\ is,\ (1-q)S_b(\varrho,\varrho,T\varrho)\leq 0.$$

Therefore we get $S_b(\varrho,\varrho,T\varrho)=0$, as $0\leq q\leq\frac{1}{s^2}$. Hence T$\varrho=\varrho$.
Thus, ϱ is a fixed point of T. Now, we claim that ϱ is unique.
For this, let θ is any other fixed point of T. It follows from inequality (5.2.1) and Lemma 1.3.3., we get

$$\begin{aligned}S_b(\varrho,\varrho,\theta) &= S_b(T\varrho,T\varrho,T\theta)\\ &\leq \psi(S_b(\varrho,\varrho,\theta),S_b(\varrho,\varrho,T\varrho),S_b(\theta,\theta,T\theta),\\ &S_b(\varrho,\varrho,T\varrho),\frac{1}{2s}[S_b(\varrho,\varrho,T\varrho)+S_b(\theta,\theta,T\varrho)])\\ &= \psi(S_b(\varrho,\varrho,\theta),S_b(\varrho,\varrho,\varrho),S_b(\theta,\theta,\theta),\\ &S_b(\varrho,\varrho,\varrho),\frac{1}{2s}[S_b(\varrho,\varrho,\varrho)+S_b(\theta,\theta,\varrho)])\\ &\leq \psi(S_b(\varrho,\varrho,\theta),0,0,0,\frac{1}{2}S_b(\varrho,\varrho,\theta))\end{aligned}$$

Since $\psi\in\Psi$ holds the property (R3), we obtain

$$S_b(\varrho,\varrho,\theta)\leq qS_b(\varrho,\varrho,\theta)$$

$$that\ is,\ (1-q)S_b(\varrho,\varrho,\theta)\leq 0.$$

Therefore we get $S_b(\varrho,\varrho,\theta)=0$, as $0\leq q\leq\frac{1}{s^2}$. Hence $\varrho=\theta$.
Thus the fixed point of T is unique.

5.2.2 Corollary: Let T : X $\to$ X be a function on a complete S_b-metric space (X, S_b) and let T satisfy $S_b(T\xi, T\vartheta, Tw) \leq \gamma S_b(\xi, \vartheta, w)$ for all $\xi, \vartheta, w \in$ X, where $\gamma \in$ [0, 1) is a constant. Then T has one and only one fixed point in X. Moreover, T is continuous at the fixed point.

Proof: We can prove easily from Theorem 5.2.1. with $\psi(a, b, c, d, e) = \gamma a$, for $\gamma \in [0, 1)$ and a,b,c,d,e $\in \mathbb{R}_+$.

5.2.3 Example: Let (X, S_b) be a complete S_b-metric space with s=4, where X = [0, 1] and $S_b(\xi, \vartheta, w) = (|\xi - w| + |\vartheta - w|)^2$.

Now, we let the mapping T: X $\to$ X be defined by $T(\xi) = \frac{\xi}{5}$, $\forall \xi \in$ [0, 1].

Then
$$\begin{aligned} S_b(T\xi, T\vartheta, Tw) &= (|T\xi - Tw| + |T\vartheta - Tw|)^2 \\ &= (|\tfrac{\xi}{5} - \tfrac{w}{5}| + |\tfrac{\vartheta}{5} - \tfrac{w}{5}|)^2 \\ &= \tfrac{1}{25}(|\xi - w| + |\vartheta - w|)^2 \\ &\leq \tfrac{1}{25} S_b(\xi, \vartheta, w) \\ &= \gamma S_b(\xi, \vartheta, w). \end{aligned}$$
where $\gamma = \frac{1}{25} < 1$. Clearly T holds all the properties of Corollary 5.2.2. and hence $0 \in X$ is the one and only one fixed point of T.

5.3 Common fixed point results in S_b-metric spaces

Through this section, we prove a common fixed point theorem using an implicit relation in S_b-metric spaces. Further, we extend these results to a family of continuous maps and also we provide a corollary.

5.3.1 Theorem: Let T_1 and T_2 be two self maps on a complete S_b-metric space (X, S_b) with s $\geq$ 1 and

$$\begin{aligned} S_b(T_1\xi, T_1\vartheta, T_2w) \leq \psi(&S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T_1\xi), S_b(w, w, T_2w), \\ &S_b(\xi, \xi, T_1\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1\vartheta) + S_b(w, w, T_1\xi)]) \qquad (5.3.1) \end{aligned}$$

for all $\xi, \vartheta, w \in X$ and $\psi \in \Psi$. Then T_1 and T_2 have a unique common fixed point in X, whenever ψ holds the conditions (R1),(R2) and (R3).

Proof: Consider an arbitrary $\xi_0 \in X$ and a sequence $\{\xi_\ell\}$ in X defined by $\xi_{2\ell+1} = T_1\xi_{2\ell}$ and $\xi_{2\ell+2} = T_2\xi_{2\ell+1}$, for ℓ = 0,1,2,3.....

It follows from inequality (5.3.1.) and Lemma 1.3.3., that

$$\begin{aligned}
S_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) &= S_b(T_1\xi_{2\ell}, T_1\xi_{2\ell}, T_2\xi_{2\ell-1}) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}), S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, T_2\xi_{2\ell-1}), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}) + S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, T_1\xi_{2\ell})]) \\
&= \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell}), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}) + S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell+1})]) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), sS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), \\
&sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \frac{1}{2s}[sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) \\
&+ 2sS_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell}) + sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell})]) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \\
&sS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \\
&\frac{1}{2s}[2sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) + 2s^2S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1})]) \qquad (5.3.2.)
\end{aligned}$$

Since $\psi \in \Psi$ holds the property (R1), we can find q $\in [0, \frac{1}{s^2}]$ such that

$$S_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) \leq qS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}) \leq q^{2\ell}S_b(\xi_1, \xi_1, \xi_0) \qquad (5.3.3.)$$

For $\ell, m \in \mathbb{N}$ with $\ell <$ m, From the equation (5.3.3.) and Lemma 1.3.3., we get

$$\begin{aligned}
S_b(\xi_\ell, \xi_\ell, \xi_m) &\leq 2sS_b(\xi_{\ell-1}, \xi_\ell, \xi_{\ell+1}) + s^2S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_m) \\
&\leq 2sS_b(\xi_\ell, \xi_\ell, \xi_{\ell+1}) + s^2[2S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_{\ell+2}) \\
&+ s^2S_b(\xi_{\ell+2}, \xi_{\ell+2}, \xi_m)] \\
&\leq 2sq^\ell[1 + s^2q + (s^2q)^2 + ...]S_b(\xi_0, \xi_0, \xi_1) \\
&\leq (\frac{2sq^\ell}{1 - s^2q})S_b(\xi_0, \xi_0, \xi_1)
\end{aligned}$$

Taking the limit as $\ell \to \infty$, we get $S_b(\xi_\ell, \xi_\ell, \xi_m) \to 0$, since q $\in [0, \frac{1}{s^2}]$ and $s \geq 1$. This claims that the sequence $\{\xi_\ell\}$ becomes a Cauchy in X and since X is complete, we can find a $\varrho \in$ X so that $\lim_{\ell\to\infty} \xi_\ell = \varrho$. Now we claim that ϱ is a common fixed point of T_1 and T_2. Now, we consider,

$$\begin{aligned}
S_b(\xi_{2\ell+1}, \xi_{2\ell+1}, T_1\varrho) &= S_b(T_1\xi_{2\ell}, T_1\xi_{2\ell}, T_1\varrho) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \varrho), S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}), S_b(\varrho, \varrho, T_1\varrho), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, T_1\xi_{2\ell}) + S_b(\varrho, \varrho, T_1\xi_{2\ell})]) \\
&= \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \varrho), S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), S_b(\varrho, \varrho, T_1\varrho), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}) + S_b(\varrho, \varrho, \xi_{2\ell+1})]) \qquad (5.3.4)
\end{aligned}$$

Letting $\ell \to \infty$, we get

$$\begin{aligned} S_b(\varrho, \varrho, T_1\varrho) &\leq \psi(S_b(\varrho, \varrho, \varrho), S_b(\varrho, \varrho, \varrho), S_b(\varrho, \varrho, T_1\varrho), \\ &\qquad S_b(\varrho, \varrho, \varrho), \frac{1}{2s}[S_b(\varrho, \varrho, \varrho) + S_b(\varrho, \varrho, \varrho)]) \\ \textit{that is,}\ \ S_b(\varrho, \varrho, T_1\varrho) &\leq \psi(0, 0, S_b(\varrho, \varrho, T_1\varrho), 0, 0) \end{aligned}$$

Since $\psi \in \Psi$ holds the property(R2), we have

$$\begin{aligned} S_b(\varrho, \varrho, T_1\varrho) &\leq q S_b(\varrho, \varrho, T_1\varrho) \\ \textit{that is,}\ \ (1-q)S_b(\varrho, \varrho, T_1\varrho) &\leq 0. \end{aligned}$$

Therefore we get $S_b(\varrho, \varrho, T_1\varrho) = 0$, as $0 \leq q \leq \frac{1}{s^2}$. Hence $T_1\varrho = \varrho$.
Similarly, we can show that $T_2\varrho = \varrho$. Therefore, ϱ is a common fixed point of T_1 and T_2. Now, we prove that ϱ is one and only one common fixed point. For this, suppose θ is another common fixed point of T_1 and T_2. Utilizing the Lemma 1.3.3. and equation (5.3.1.), we have

$$\begin{aligned} S_b(\varrho, \varrho, \theta) &= S_b(T_1\varrho, T_1\varrho, T_2\theta) \\ &\leq \psi(S_b(\varrho, \varrho, \theta), S_b(\varrho, \varrho, T_1\varrho), S_b(\theta, \theta, T_2\theta), \\ &\quad S_b(\varrho, \varrho, T_1\varrho), \frac{1}{2s}[S_b(\varrho, \varrho, T_1\varrho) + S_b(\theta, \theta, T_1\varrho)]) \\ &= \psi(S_b(\varrho, \varrho, \theta), S_b(\varrho, \varrho, \varrho), S_b(\theta, \theta, \theta), \\ &\quad S_b(\varrho, \varrho, \varrho), \frac{1}{2s}[S_b(\varrho, \varrho, \varrho) + S_b(\theta, \theta, \varrho)]) \\ &\leq \psi(S_b(\varrho, \varrho, \theta), 0, 0, 0, \frac{1}{2}S_b(\varrho, \varrho, \theta)) \end{aligned}$$

Since $\psi \in \Psi$ holds the property(R3), we have

$$\begin{aligned} S_b(\varrho, \varrho, \theta) &\leq q S_b(\varrho, \varrho, \theta) \\ \textit{that is,}\ \ (1-q)S_b(\varrho, \varrho, \theta) &\leq 0. \end{aligned}$$

Therefore, we get $S_b(\varrho, \varrho, \theta) = 0$, as $0 \leq q \leq \frac{1}{s^2}$. Hence $\varrho = \theta$. Therefore ϱ is the unique common fixed point of T_1 and T_2.

5.3.2 Theorem: Let $T_1, T_2 \colon \mathrm{X} \to \mathrm{X}$ be two continuous functions on a complete S_b-metric space (X, S_b) with s ≥ 1 and

$$\begin{aligned} S_b(T_1^p\xi, T_1^p\vartheta, T_2^q w) &\leq \psi(S_b(\xi, \vartheta, w), S_b(\vartheta, \vartheta, T_1^p\xi), S_b(w, w, T_2^q w), \\ &\quad S_b(\xi, \xi, T_1^p\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1^p\vartheta) + S_b(w, w, T_1^p\xi)]) \end{aligned} \tag{5.3.5}$$

for all $\xi, \vartheta, w \in$ X, where $\psi \in \Psi$ and p , q$\in \mathbb{Z}$. Then T_1 and T_2 have a unique common fixed point in X, whenever ψ satisfies the conditions (R1),(R2) and (R3).

Proof: Since T_1^p and T_2^q satisfy the conditions of Theorem 5.3.1. So, T_1^p and T_2^q have one and only one common fixed point. Let λ be the common fixed point.

Then, we have $T_1^p\lambda = \lambda \Rightarrow T_1(T_1^p\lambda) = T_1\lambda \Rightarrow T_1^p(T_1\lambda) = T_1\lambda$.

If $T_1\lambda = \lambda_0$, then $T_1^p\lambda_0 = \lambda_0$. So, $T_1\lambda$ is a fixed point of T_1^p.

Similarly, $T_2(T_2^q\lambda) = T_2^q(T_2\lambda) = T_2\lambda$. Now, utilizing equation (5.3.5) and Lemma 1.3.3., we obtain

$$\begin{aligned} S_b(\lambda,\lambda,T_1\lambda) &= S_b(T_1^p\lambda, T_1^p\lambda, T_1^p(T_1\lambda)) \\ &\le \psi(S_b(\lambda,\lambda,T_1\lambda), S_b(\lambda,\lambda,T_1^p\lambda), S_b(T_1\lambda,T_1\lambda,T_1^p(T_1\lambda)), \\ &S_b(\lambda,\lambda,T_1^p\lambda), \frac{1}{2s}[S_b(\lambda,\lambda,T_1^p\lambda) + S_b(T_1\lambda,T_1\lambda,T_1^p\lambda)]) \\ &= \psi(S_b(\lambda,\lambda,T_1\lambda), S_b(\lambda,\lambda,\lambda), S_b(T_1\lambda,T_1\lambda,T_1\lambda), \\ &S_b(\lambda,\lambda,\lambda), \frac{1}{2s}[S_b(\lambda,\lambda,\lambda) + S_b(T_1\lambda,T_1\lambda,\lambda)]) \\ &\le \psi(S_b(\lambda,\lambda,T_1\lambda), 0, 0, 0, \frac{1}{2}[S_b(\lambda,\lambda,T_1\lambda)]) \end{aligned}$$

Since $\psi \in \Psi$ holds the property (R3), we get

$$S_b(\lambda,\lambda,T_1\lambda) \le qS_b(\lambda,\lambda,T_1\lambda)$$
$$\textit{that is},\ (1-q)S_b(\lambda,\lambda,T_1\lambda) \le 0.$$

Therefore we get $S_b(\lambda,\lambda,T_1\lambda) = 0$, as $0 \le q \le \frac{1}{s^2}$ and $s \ge 1$. Hence $T_1\lambda = \lambda$. Similarly, we can show that $T_2\lambda = \lambda$. Thus λ is a common fixed point of T_1 and T_2. To prove uniqueness of λ, let $\lambda^* \ne \lambda$ be any other common fixed point of T_1 and T_2. Then obviously λ^* is also a common fixed point of T_1^p and T_2^q, which implies $\lambda = \lambda^*$. Therefore T_1 and T_2 have one and only one common fixed point.

5.3.3 Theorem: Let $\{G_\alpha\}$ be a collection of continuous self mappings on a complete S_b-metric space (X, S_b) with s ≥ 1 and

$$\begin{aligned} S_b(G_\alpha\xi, G_\alpha\vartheta, G_\beta w) \le \psi(&S_b(\xi,\vartheta,w), S_b(\vartheta,\vartheta,G_\alpha\xi), S_b(w,w,G_\beta w), \\ &S_b(\xi,\xi,G_\alpha\vartheta), \frac{1}{2s}[S_b(\vartheta,\vartheta,G_\alpha\vartheta) + S_b(w,w,G_\alpha\xi)]) \qquad (5.3.6.) \end{aligned}$$

for all $\xi, \vartheta, w \in X$, and $\alpha, \beta \in \Psi$ with $\alpha \ne \beta$. Then there is one and only one $\varrho \in X$ satisfying $G_\alpha\varrho = \varrho$, for all $\alpha \in \Psi$.

Proof: Consider an arbitrary $\xi_0 \in$ X and a sequence $\{\xi_\ell\}$ in X defined by

$\xi_{2\ell+1} = G_\alpha \xi_{2\ell}$ and $\xi_{2\ell+2} = G_\beta \xi_{2\ell+1}$, for ℓ=0,1,2,3.....

It follows from inequality (5.3.6.) and Lemma 1.3.3., that

$$\begin{aligned}
S_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) &= S_b(G_\alpha \xi_{2\ell}, G_\alpha \xi_{2\ell}, G_\beta \xi_{2\ell-1}) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), S_b(\xi_{2\ell}, \xi_{2\ell}, G_\alpha \xi_{2\ell}), S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, G_\beta \xi_{2\ell-1}), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, G_\alpha \xi_{2\ell}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, G_\alpha \xi_{2\ell}) + S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, G_\alpha \xi_{2\ell})]) \\
&= \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell}), \\
&S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}), \frac{1}{2s}[S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell+1}) + S_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell+1})]) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), sS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), \\
&sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \frac{1}{2s}[sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) \\
&+ 2sS_b(\xi_{2\ell-1}, \xi_{2\ell-1}, \xi_{2\ell}) + sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell})]) \\
&\leq \psi(S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \\
&sS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}), sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}), \\
&\frac{1}{2s}[2sS_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) + 2s^2 S_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1})]) \quad (5.3.7.)
\end{aligned}$$

Since $\psi \in \Psi$ holds the property (R1), we can find q $\in [0, \frac{1}{s^2}]$ so that

$$S_b(\xi_{2\ell+1}, \xi_{2\ell+1}, \xi_{2\ell}) \leq qS_b(\xi_{2\ell}, \xi_{2\ell}, \xi_{2\ell-1}) \leq q^{2\ell} S_b(\xi_1, \xi_1, \xi_0) \qquad (5.3.8.)$$

For $\ell, m \in \mathbb{N}$ with m $> \ell$, by utilizing equation (5.3.8.) and Lemma 1.3.3., we have

$$\begin{aligned}
S_b(\xi_\ell, \xi_\ell, \xi_m) &\leq 2sS_b(\xi_{\ell+1}, \xi_\ell, \xi_{\ell+1}) + s^2 S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_m) \\
&\leq 2sS_b(\xi_\ell, \xi_\ell, \xi_{\ell+1}) + s^2[2S_b(\xi_{\ell+1}, \xi_{\ell+1}, \xi_{\ell+2}) + s^2 S_b(\xi_{\ell+2}, \xi_{\ell+2}, \xi_m)] \\
&\leq 2sq^\ell[1 + s^2 q + (s^2 q)^2 + ...]S_b(\xi_0, \xi_0, \xi_1) \\
&\leq (\frac{2sq^\ell}{1 - s^2 q})S_b(\xi_0, \xi_0, \xi_1)
\end{aligned}$$

Taking the limit as $\ell \to \infty$, we get $S_b(\xi_\ell, \xi_\ell, \xi_m) \to 0$, since q $\in [0, \frac{1}{s^2}]$ and $s \geq 1$. Hence the sequence $\{\xi_\ell\}$ becomes Cauchy in X and since X is complete, we can find a $\varrho \in$ X so that $\lim_{\ell \to \infty} \xi_\ell = \varrho$. It is clear that $G_\alpha \varrho = G_\beta \varrho = \varrho$, since by the continuity of G_α and G_β. Hence ϱ is a common fixed point of G_α and G_β, for any $\alpha \in \Psi$. To verify the uniqueness, consider another common fixed point θ of G_α

and G_β, where $\varrho \neq \theta$. Then utilizing Lemma 1.3.3. and equation (5.3.6.), we get

$$\begin{aligned}S_b(\varrho,\varrho,\theta) &= S_b(G_\alpha\varrho, G_\alpha\varrho, G_\beta\theta)\\ &\leq \psi(S_b(\varrho,\varrho,\theta), S_b(\varrho,\varrho,G_\alpha\varrho), S_b(\theta,\theta,G_\beta\theta),\\ &S_b(\varrho,\varrho,G_\alpha\varrho), \frac{1}{2s}[S_b(\varrho,\varrho,G_\alpha\varrho) + S_b(\theta,\theta,G_\alpha\varrho)])\\ &= \psi(S_b(\varrho,\varrho,\theta), S_b(\varrho,\varrho,\varrho), S_b(\theta,\theta,\theta),\\ &S_b(\varrho,\varrho,\varrho), \frac{1}{2s}[S_b(\varrho,\varrho,\varrho) + S_b(\theta,\theta,\varrho)])\\ &\leq \psi(S_b(\varrho,\varrho,\theta), 0, 0, 0, \frac{1}{2}S_b(\varrho,\varrho,\theta))\end{aligned}$$

Since $\psi \in \Psi$ holds the condition(R3), we get

$$\begin{aligned}S_b(\varrho,\varrho,\theta) &\leq qS_b(\varrho,\varrho,\theta)\\ \textit{that is},\ (1-q)S_b(\varrho,\varrho,\theta) &\leq 0.\end{aligned}$$

Therefore we get $S_b(\varrho,\varrho,\theta) = 0$, as $0 \leq q \leq \frac{1}{s^2}$. Hence $\varrho = \theta$. Thus ϱ is the one and only one common fixed point of G_α, $\forall \alpha \in \Psi$.

5.3.4 Corollary: Suppose that the functions T_1, T_2: X $\to$ X are two functions defined on a complete S_b-metric space (X, S_b) and satisfy $S_b(T_1\xi, T_1\vartheta, T_2w) \leq \delta S_b(\xi,\vartheta,w)$ for all $\xi,\vartheta,w \in$ X and $\delta \in [0,\ 1)$. Then T_1 and T_2 have one and only one common fixed point in X.

Proof: Follows from Theorem 5.3.1., by substituting $\psi(a,b,c,d,e) = \delta a$, for $\delta \in [0,1)$.

Chapter 6

Common and Coupled fixed point results in bicomplex valued metric spaces using CLR - properties

6.1 Introduction

In this chapter, we establish two unique common fixed point theorems for four self-mappings and six self-mappings and a common coupled fixed point theorem in a bicomplex valued metric space. Firstly, we prove a common fixed point theorem for four self-mappings by using weaker conditions such as weakly compatibility, generalized contraction and CLR_{AB} property. Then, secondly, we derive a common fixed point theorem for six self-mappings with the help of weakly compatibility and inclusion relations by using the generalized contraction. Finally, we prove a common coupled fixed point theorem in the bicomplex valued metric space. The aforementioned findings are extensions and generalizations of Iqbal H.Jebril, S. Kumar Datta, Rakesh Sarkar and N. Biswas [108].

6.2 Main Results and Examples

Now, we derive a common fixed point theorem to four self-mappings using weakly compatibility and CLR_{AB} property in this section. Further, we also give a corollary and an example to support the result.

6.2.1 Theorem: Suppose (X, d) is a complete bicomplex valued metric space and h,k,A and B are self mappings on X satisfying

(i) $d(h\varpi, k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi, B\vartheta) + \tau_2 d(A\varpi, h\varpi) + \tau_3 d(B\vartheta, k\vartheta)$, $\forall \varpi, \vartheta \in X$, where τ_2,τ_1 and τ_3 are non negative reals such that $1 > \tau_1 + \tau_2 + \tau_3$.

(ii) $\{B, k\}$ and $\{A, h\}$ are weakly compatible,

(iii) $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property.

Then h,k,A and B have one and only one common fixed point.

Proof: Since $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property, we can find sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in X so that

$\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = \jmath$,

for some $\jmath \in$ AX∩BX. Then $\jmath =$ Bη_1 = Aη_2, for some $\eta_1, \eta_2 \in X$.

Now we claim that kη_1= Bη_1. To each n∈ ℕ, we have

$d(h\varpi_n, k\eta_1) \preceq_{i_2} \tau_1 d(A\varpi_n, B\eta_1) + \tau_2 d(A\varpi_n, h\varpi_n) + \tau_3 d(B\eta_1, k\eta_1)$

Letting n $\to \infty$, we get

$d(B\eta_1, k\eta_1) \preceq_{i_2} \tau_1 d(B\eta_1, B\eta_1) + \tau_2 d(B\eta_1, B\eta_1) + \tau_3 d(B\eta_1, k\eta_1)$

i.e.,$d(B\eta_1, k\eta_1) \preceq_{i_2} \tau_3 d(B\eta_1, k\eta_1)$

Therefore we have

$\|d(B\eta_1, k\eta_1)\| \leq \tau_3 \ \|d(B\eta_1, k\eta_1)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore we get $\|d(B\eta_1, k\eta_1)\| = 0$. Thus $\mathrm{B}\eta_1 = k\eta_1$.

Now we claim that $\mathrm{A}\eta_2 = h\eta_2$. To each n$\in \mathbb{N}$, we consider

$d(h\eta_2, k\vartheta_n) \preceq_{i_2} \tau_1 d(A\eta_2, B\vartheta_n) + \tau_2 d(A\eta_2, h\eta_2) + \tau_3 d(B\vartheta_n, k\vartheta_n)$

Letting n $\to \infty$, we get

$d(h\eta_2,\mathrm{A}\eta_2) \preceq_{i_2} \tau_1 d(A\eta_2, A\eta_2) + \tau_2 d(A\eta_2, h\eta_2) + \tau_3 d(A\eta_2, A\eta_2)$

i.e., $d(h\eta_2,\mathrm{A}\eta_2) \preceq_{i_2} \tau_2 d(h\eta_2,\mathrm{A}\eta_2)$

Therefore we have $\|d(h\eta_2,\mathrm{A}\eta_2)\| \leq \tau_2\|d(h\eta_2,\mathrm{A}\eta_2)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|d(h\eta_2,\mathrm{A}\eta_2)\| = 0$. Thus $h\eta_2 = A\eta_2$.

Hence $B\eta_1 = k\eta_1 = h\eta_2 = A\eta_2 = \jmath$.

Given that $\{A, h\}$ is weakly compatible and $\mathrm{h}\eta_2 = \mathrm{A}\eta_2$. We get $\mathrm{hA}\eta_2 = \mathrm{Ah}\eta_2$. So, $\mathrm{h}\jmath$=$\mathrm{A}\jmath$.

Given that $\{B, k\}$ is weakly compatible and $\mathrm{k}\eta_1 = \mathrm{B}\eta_1$. We get $\mathrm{kB}\eta_1 = \mathrm{Bk}\eta_1$. So, $\mathrm{k}\jmath = \mathrm{B}\jmath$.

Now we prove that $\mathrm{h}\jmath = \jmath$.

Consider

$d(h\jmath,\mathrm{k}\eta_1) \preceq_{i_2} \tau_1 d(A\jmath,\mathrm{B}\eta_1) + \tau_2 d(A\jmath,\mathrm{h}\jmath) + \tau_3 d(B\eta_1,\mathrm{k}\eta_1)$

i.e., $d(h\jmath,\jmath) \preceq_{i_2} \tau_1 d(h\jmath,\jmath) + \tau_2 d(h\jmath,\mathrm{h}\jmath) + \tau_3 d(\jmath,\jmath)$

i.e., $d(h\jmath,\jmath) \preceq_{i_2} \tau_1 d(h\jmath,\jmath)$

Therefore we have $\|d(h\jmath,\jmath)\| \leq \tau_1 \ \|d(h\jmath,\jmath)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|d(h\jmath,\jmath)\| = 0$. Thus $\mathrm{h}\jmath = \jmath$. So, we have $\mathrm{h}\jmath = \jmath = \mathrm{A}\jmath$.

Now we prove that $\mathrm{k}\jmath = \jmath$.

Consider

$d(h\eta_2,\mathrm{k}\jmath) \preceq_{i_2} \tau_1 d(A\eta_2,\mathrm{B}\jmath) + \tau_2 d(A\eta_2,\mathrm{h}\eta_2) + \tau_3 d(B\jmath,\mathrm{k}\jmath)$

i.e., $d(\jmath,\mathrm{k}\jmath) \preceq_{i_2} \tau_1 d(\jmath,\mathrm{k}\jmath) + \tau_2 d(\jmath,\jmath) + \tau_3 d(k\jmath,\mathrm{k}\jmath)$

Therefore we have $\|d(\jmath,\mathrm{k}\jmath)\| \leq \tau_1\|d(\jmath,\mathrm{k}\jmath)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|d(\jmath,\mathrm{k}\jmath)\| = 0$. Thus $\mathrm{k}\jmath = \jmath$. So, we have $\mathrm{k}\jmath = \jmath = \mathrm{B}\jmath$.

Hence $\mathrm{h}\jmath = \mathrm{A}\jmath = \jmath = \mathrm{k}\jmath = \mathrm{B}\jmath$.

Therefore $\jmath$ is common fixed point of A,h,k and B.

Now we prove $\jmath$ is unique.

For this, we consider another common fixed point δ of h,k,A and B.

Then hδ = kδ = Aδ = Bδ = δ.

Now, consider

$d(\jmath,\delta) = d(h\jmath,\text{k}\delta) \preceq_{i_2} \tau_1 d(A\jmath,\text{B}\delta) + \tau_2 d(A\jmath,\text{h}\jmath) + \tau_3 d(B\delta,\text{k}\delta)$

i.e., $d(\jmath,\delta) \preceq_{i_2} \tau_1 d(\jmath,\delta) + \tau_2 d(\jmath,\jmath) + \tau_3 d(\delta,\delta)$

i.e., $d(\jmath,\delta) \preceq_{i_2} \tau_1\ d(\jmath,\delta)$

Therefore we have $\|d(\jmath,\delta)\| \leq \tau_1 \|d(\jmath,\delta)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Hence, we get $\|d(\jmath,\delta)\| = 0$.Thus $\jmath = \delta$.

Hence, $\jmath$ is the one and only one common fixed point of h,k,A and B.

6.2.2 Example: Suppose X = [0, 1] and define d: X × X → C_2 by

$$d(\varpi,\vartheta) = \begin{cases} 0, & \text{for } \varpi = \vartheta \ \ and \\ i_2 max\{\varpi,\vartheta\}, & otherwise \end{cases}$$

for all $\varpi, \vartheta \in X$.

Define h,k,A and B be self maps on X as:

For $\varpi \in X, h(\varpi) = \frac{\varpi}{3}, k(\varpi) = \frac{\varpi}{3}, A(\varpi) = \varpi$ and $B(\varpi) = \varpi$.

First, we show that {h,A} and {k,B} satisfy CLR_{AB} property. For this, we choose $\varpi_n = \frac{1}{2n}$ and $\vartheta_n = \frac{1}{3n+1}$ for n$\in$ **N**. Clearly, $<\varpi_n>$ and $<\vartheta_n>$ are in X. Then $d(A\varpi_n, 0) = d(\frac{1}{2n}, 0) \to 0$ as $n \to \infty$. Also, $d(h\varpi_n, 0) = d(\frac{1}{6n}, 0)$ $\to$ to 0 as $n \to \infty$. Similarly, we get $d(k\vartheta_n, 0) = d(\frac{1}{9n+1}, 0) \to 0$ as $n \to \infty$ and $d(B\vartheta_n, 0) = d(\frac{1}{3n+1}, 0) \to 0$ as $n \to \infty$. Since A0 = 0 = B0. So, we have 0 ∈ AX ∩ BX. Therefore, we have sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in X so that $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = 0$. Thus {h,A} and {k,B} satisfy CLR_{AB} property.

Now we show that {h,A} and {k,B} are weakly compatible. Now, $h\varpi = A\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and hA(0) = h(0) = 0 and Ah(0) = A(0) = 0. Thus $hA(\varpi) = Ah(\varpi)$, whenever $h\varpi = A\varpi$, for all $\varpi \in X$. Hence {h,A} is weakly compatible in X.

Also, $k\varpi = B\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and kB(0) = Bk(0). Thus, $kB(\varpi) = Bk(\varpi)$, whenever $k\varpi = B\varpi$ for all $\varpi \in X$. Hence, {k,B} is weakly compatible in X.

Finally, we show that condition (i) of the Theorem holds. Now, $d(h\varpi, k\vartheta) = d(\frac{\varpi}{3}, \frac{\vartheta}{3}) = i_2 max\{\frac{\varpi}{3}, \frac{\vartheta}{3}\}$,

$d(A\varpi, B\vartheta) = d(\varpi,\vartheta) = i_2 max\{\varpi,\vartheta\}$,

$d(A\varpi, h\varpi) = d(\varpi, \frac{\varpi}{3}) = i_2\text{max}\{\varpi, \frac{\varpi}{3}\} = i_2\varpi,$

$d(B\vartheta, k\vartheta) = d(\vartheta, \frac{\vartheta}{3}) = i_2 max\{\vartheta, \frac{\vartheta}{3}\} = i_2\vartheta.$

Case(a) if $\varpi > \vartheta$, then

$d(h\varpi, k\vartheta) = i_2 max\{\frac{\varpi}{3}, \frac{\vartheta}{3}\} = i_2\frac{\varpi}{3},$

$d(A\varpi, B\vartheta) = i_2 max\{\varpi, \vartheta\} = i_2\varpi,$

$d(A\varpi, h\varpi) = i_2 max\{\varpi, \frac{\varpi}{3}\} = i_2\varpi,$

$d(B\vartheta, k\vartheta) = i_2\vartheta.$

Now, $d(h\varpi, k\vartheta) = i_2\frac{\varpi}{3} \preceq_{i_2} \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$

i.e., $d(h\varpi, k\vartheta) \preceq_{i_2} \frac{1}{4}d(A\varpi, B\vartheta) + \frac{1}{4}d(A\varpi, h\varpi) + \frac{1}{4}d(B\vartheta, k\vartheta)$

By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$, Here τ_1,τ_2,τ_3 be non negative reals so that $1 > \tau_1 + \tau_2 + \tau_3$. Hence

$d(h\varpi, k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi, B\vartheta) + \tau_2 d(A\varpi, h\varpi) + \tau_3 d(B\vartheta, k\vartheta).$

Case(b) if $\varpi < \vartheta$, then

$d(h\varpi, k\vartheta) = i_2\text{max}\{\frac{\varpi}{3}, \frac{\vartheta}{3}\} = i_2\frac{\vartheta}{3},$

$d(A\varpi, B\vartheta) = i_2 max\{\varpi, \vartheta\} = i_2\vartheta,$

$d(A\varpi, h\varpi) = i_2 max\{\varpi, \frac{\varpi}{3}\} = i_2\varpi,$

$d(B\vartheta, k\vartheta) = i_2\vartheta.$

Now, $d(h\varpi, k\vartheta) = i_2\frac{\vartheta}{3} \preceq_{i_2} \frac{1}{4}[i_2\vartheta] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$

i.e., $d(h\varpi, k\vartheta) \preceq_{i_2} \frac{1}{4}d(A\varpi, B\vartheta) + \frac{1}{4}d(A\varpi, h\varpi) + \frac{1}{4}d(B\vartheta, k\vartheta)$

By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$,

Here τ_2,τ_1,τ_3 be non negative real numbers so that $\tau_1 + \tau_2 + \tau_3 < 1$. Hence

$d(h\varpi, k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi, B\vartheta) + \tau_2 d(A\varpi, h\varpi) + \tau_3 d(B\vartheta, k\vartheta).$

Therefore 0 in X is the unique common fixed point of h, k, A and B.

6.2.3 Corollary: Suppose (X, d) is a complete bicomplex valued metric space and h,k and A are self mappings on X satisfy

(i) $d(hz, kw) \preceq_{i_2} \tau_1 d(Az, Aw) + \tau_2 d(Az, hz) + \tau_3 d(Aw, kw)$, for all z,w $\in X$, where τ_2,τ_1 and τ_3 are non negative reals such that $1 > \tau_1 + \tau_2 + \tau_3$.

(ii) $\{h, A\}$ and $\{k, A\}$ are weakly compatible,

(iii) $\{h, A\}$ and $\{k, A\}$ satisfy CLR_A property.

Then h,k and A have one and only one common fixed point.

Proof: We can prove this result easily by substituting B = A in the Theorem 6.2.1.

6.3 Common fixed point results for Six maps using weakly compatibility

Now, we establish a common fixed point theorem for six self-mappings with the help of weakly compatibility and inclusion relations and by defining the generalized contraction. Moreover, we deduce a corollary from it.

6.3.1 Theorem: Suppose (X, d) is a complete bicomplex valued metric space and that H,I,C,P,Q,R are the self mappings on X satisfy (i) H(X) $\supseteq$ QR(X) and I(X) $\supseteq$ CP(X) (ii) $d(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1 d(H\varpi, I\vartheta) + \tau_2 d(H\varpi, CP\varpi) + \tau_3 d(I\vartheta, QR\vartheta)$ $+ \tau_4 d(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in X$, where τ_1,τ_2,τ_3 and τ_4 be non negative real numbers such that $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$. (iii) Suppose (QR,I) and (CP,H) are weakly compatible and (iv) (Q,R), (Q,I) (R,I),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P,I and H contains one and only one common fixed point in X.

Proof: Let $\varpi_0 \in$ X. Since H(X) $\supseteq$ QR(X) and I(X) $\supseteq$ CP(X), we can find a sequence $\{\varpi_n'\}$ in X such that
CPϖ_{2l} = I$\varpi_{2l+1} = \varpi_{2l}'$ and QRϖ_{2l+1} = H$\varpi_{2l+2} = \varpi_{2l+1}'$ for l=0,1,2,....
Consider,

$$\begin{aligned} d(\varpi_{2l}', \varpi_{2l+1}') &= d(CP\varpi_{2l}, QR\varpi_{2l+1}) \\ &\preceq_{i_2} \tau_1 d(H\varpi_{2l}, I\varpi_{2l+1}) + \tau_2 d(H\varpi_{2l}, CP\varpi_{2l}) \\ &+ \tau_3 d(I\varpi_{2l+1}, QR\varpi_{2l+1}) + \tau_4 d(H\varpi_{2l}, QR\varpi_{2l+1}) \\ &= \tau_1 d(\varpi_{2l-1}', \varpi_{2l}') + \tau_2 d(\varpi_{2l-1}', \varpi_{2l}') + \tau_3 d(\varpi_{2l}', \varpi_{2l+1}') \\ &+ \tau_4 d(\varpi_{2l-1}', \varpi_{2l+1}') \\ &= \tau_1 d(\varpi_{2l-1}', \varpi_{2l}') + \tau_2 d(\varpi_{2l-1}', \varpi_{2l}') + \tau_3 d(\varpi_{2l}', \varpi_{2l+1}') \\ &+ \tau_4 [d(\varpi_{2l-1}', \varpi_{2l}') + d(\varpi_{2l}', \varpi_{2l+1}')] \end{aligned}$$

i.e., $(1 - \tau_3 - \tau_4)\ d(\varpi_{2l}', \varpi_{2l+1}') \preceq_{i_2} (\tau_1 + \tau_2 + \tau_4)\ d(\varpi_{2l-1}', \varpi_{2l}')$
i.e., $d(\varpi_{2l}', \varpi_{2l+1}') \preceq_{i_2} (\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4})\ d(\varpi_{2l-1}', \varpi_{2l}')$

Similarly, we consider

$$\begin{aligned}
d(\varpi'_{2l+1}, \varpi'_{2l+2}) &= d(QR\varpi_{2l+1}, CP\varpi_{2l+2})\\
&= d(CP\varpi_{2l+2}, QR\varpi_{2l+1})\\
&\preceq_{i_2} \tau_1 d(H\varpi_{2l+2}, I\varpi_{2l+1}) + \tau_2 d(H\varpi_{2l+2}, CP\varpi_{2l+2})\\
&+ \tau_3 d(I\varpi_{2l+1}, QR\varpi_{2l+1}) + \tau_4 d(H\varpi_{2l+2}, QR\varpi_{2l+1})\\
&= \tau_1 d(\varpi'_{2l+1}, \varpi'_{2l}) + \tau_2 d(\varpi'_{2l+1}, \varpi'_{2l+2}) + \tau_3 d(\varpi'_{2l}, \varpi'_{2l+1})\\
&+ \tau_4 d(\varpi'_{2l+1}, \varpi'_{2l+1})
\end{aligned}$$

i.e., (1-τ_2) $d(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\tau_1 + \tau_3) d(\varpi'_{2l}, \varpi'_{2l+1})$
i.e., $d(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\frac{\tau_1+\tau_3}{1-\tau_2}) d(\varpi'_{2l}, \varpi'_{2l+1})$
Let σ = max $\{\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4}, \frac{\tau_1+\tau_3}{1-\tau_2}\}$.
Then σ <1, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Now, for m,$l \in$ N and $l <$ m, we consider

$$\begin{aligned}
d(\varpi'_l, \varpi'_m) &\preceq_{i_2} d(\varpi'_l, \varpi'_{l+1}) + d(\varpi'_{l+1}, \varpi'_{l+2}) + ... + d(\varpi'_{m-1}, \varpi'_m)\\
&\preceq_{i_2} (\sigma^l + \sigma^{l+1} + ... + \sigma^{m-1}) d(\varpi'_0, \varpi'_1)
\end{aligned}$$

i.e., $d(\varpi'_l, \varpi'_m) \preceq_{i_2} (\frac{\sigma^l}{1-\sigma}) d(\varpi'_0, \varpi'_1)$
Therefore we obtain
$\|d(\varpi'_l, \varpi'_m)\| \preceq_{i_2} (\frac{\sigma^l}{1-\sigma}) \|d(\varpi'_0, \varpi'_1)\|$
Since $\sigma < 1$, as n,m $\to \infty$, we get $\|d(\varpi'_l, \varpi'_m)\| \to 0$
Hence $\{\varpi'_n\}$ is a Cauchy sequence in X, which is complete. So $\exists$ a $\jmath \in$ X so that $\lim_{n\to\infty} CP\varpi_{2n} = \lim_{n\to\infty} I\varpi_{2n+1} = \lim_{n\to\infty} QR\varpi_{2n+1} = \lim_{n\to\infty} P\varpi_{2n+2} = \jmath$.
Since QR(X) $\subseteq$ H(X), $\exists$ z $\in$ X such that Hz = $\jmath$.
Now we consider

$$\begin{aligned}
d(CPz, \jmath) &\preceq_{i_2} d(CPz, QR\varpi_{2n+1}) + d(QR\varpi_{2n+1}, \jmath)\\
&\preceq_{i_2} \tau_1 d(Hz, I\varpi_{2n+1}) + \tau_2 d(Hz, CPz) + \tau_3 d(I\varpi_{2n+1}, QR\varpi_{2n+1})\\
&+ \tau_4 d(Hz, QR\varpi_{2n+1}) + d(QR\varpi_{2n+1}, \jmath)
\end{aligned}$$

Letting n $\to \infty$, we have
(CPz, $\jmath$) $\preceq_{i_2} \tau_1 d(\jmath, \jmath) + \tau_2 d(\jmath, CPz) + \tau_3 d(\jmath, \eta) + \tau_4 d(\jmath, , \jmath) + d(\jmath, \jmath)$
Therefore we get
$\|d(CPz, \jmath)\| \leq \tau_2 \|d(CPz, \jmath)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore we get, $\|d(CPz, \jmath)\| = 0$.

Hence CPz = Hz = $\jmath$.

Again since, CP(X) $\subseteq$ I(X), there exists a $w \in$ X with Iw = $\jmath$.

Now we consider,

$$\begin{aligned} d(\jmath, QRw) &= d(CPz, QRw) \\ &\preceq_{i_2} \tau_1 d(Hz, Iw) + \tau_2 d(Hz, CPz) + \tau_3 d(Iw, QRw) + \tau_4 d(Hz, QRw) \end{aligned}$$

i.e., $d(\jmath, QRw) \preceq_{i_2} (\tau_3 + \tau_4) d(\jmath, QRw)$

i.e., $\|d(\jmath, QRw)\| \preceq_{i_2} (\tau_3 + \tau_4)\|d(\jmath, QRw)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.

Therefore we get, $\|d(\jmath, QRw)\| = 0$.

Hence QRw = $\jmath$ = Iw.

Thus we get CPz = Hz = QRw = Iw = $\jmath$.

Since I and QR are weakly compatible, I(QR)w = QR(I)w implies I$\jmath$ = QR$\jmath$.

Since CP and H are weakly compatible, (CP)Hz = H(CP)z implies CP$\jmath$ = H$\jmath$.

Now we show that CP$\jmath$ = H$\jmath$ = $\jmath$:

We now consider

$$\begin{aligned} d(CP\jmath, \jmath) &= d(CP\jmath, QRw) \\ &\preceq_{i_2} \tau_1 d(H\jmath, Iw) + \tau_2 d(H\jmath, CP\jmath) + \tau_3 d(Iw, QRw) + \tau_4 d(H\jmath, QRw) \\ &= \tau_1 d(CP\jmath, \jmath) + \tau_2 d(H\jmath, H\jmath) + \tau_3 d(Iw, Iw) + \tau_4 d(CP\jmath, \jmath) \end{aligned}$$

i.e., $d(CP\jmath, \jmath) \preceq_{i_2} (\tau_1 + \tau_4) d(CP\jmath, \jmath)$

i.e., $\|d(CP\jmath, \jmath)\| \leq (\tau_1 + \tau_4)\|d(CP\jmath, \jmath)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.

Therefore, we get $\|d(CP\jmath, \jmath)\| = 0$.

Hence $CP\jmath = \jmath = H\jmath$.

Now, we show that $QR\jmath = \jmath$:

We now consider

$$\begin{aligned} d(\jmath, QR\jmath) &= d(CP\jmath, QR\jmath) \\ &\preceq_{i_2} \tau_1 d(H\jmath, I\jmath) + \tau_2 d(H\jmath, CP\jmath) + \tau_3 d(I\jmath, QR\jmath) + \tau_4 d(H\eta, QR\eta) \\ &= \tau_1 d(\jmath, QR\jmath) + \tau_2 d(H\jmath, H\jmath) + \tau_3 d(I\jmath, I\jmath) + \tau_4 d(\jmath, QR\jmath) \end{aligned}$$

i.e., $d(\jmath, QR\jmath) \preceq_{i_2} (\tau_1 + \tau_4) d(\jmath, QR\jmath)$

i.e., $\|d(\jmath, QR\jmath)\| \leq (\tau_1 + \tau_4)\|d(\jmath, QR\jmath)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.

Therefore, we get $\|d(\jmath, QR\jmath)\| = 0$.

Hence, $QR\jmath = \jmath = I\jmath$.

Thus, we get $CP\jmath = H\jmath = QR\jmath = I\jmath = \jmath$.

So, $\jmath$ be a common fixed point of H,I,CP and QR.

From condition (iv) of the Theorem, we have

$Q\jmath = Q(QR\jmath) = Q(RQ\jmath) = (QR)Q\jmath$ and

$Q\jmath = Q(H\jmath) = H(Q\jmath)$;

$R\jmath = R(H\jmath) = HR\jmath$ and

$R\jmath = R(QR\jmath) = (RQ)R\jmath = (QR)R\jmath$.

Thus Q$\jmath$ and R$\jmath$ are common fixed points of (QR,H).

Therefore, we get Q$\jmath$ = $\jmath$ = R$\jmath$ = H$\jmath$ = QR$\jmath$.

Similarly, we can easily prove, C$\jmath$ = $\jmath$ = P$\jmath$ = I$\jmath$ = CP$\jmath$.

Thus, $\jmath$ is a common fixed point of H,I,C,P,Q and R.

Now we prove $\jmath$ is unique. Suppose γ is a common fixed point of H,I,C,P,Q and R other than $\jmath$.

Now we consider,

$$\begin{aligned} d(\jmath, \gamma) &= d(CP\jmath, QR\gamma) \\ &\preceq_{i_2} \tau_1 d(H\jmath, I\gamma) + \tau_2 d(H\jmath, CP\eta) + \tau_3 d(I\gamma, QR\gamma) + \tau_4 d(H\jmath, QR\gamma) \end{aligned}$$

i.e., $d(\jmath, \gamma) \preceq_{i_2} (\tau_1 + \tau_4) d(\jmath, \gamma)$

i.e.,$\|d(\jmath, \gamma)\| \preceq_{i_2} (\tau_1 + \tau_4)\|d(\jmath, \gamma)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.

Therefore, we get $\|d(\jmath, \gamma)\| = 0$.

Hence, we get $\jmath = \gamma$.

Thus $\jmath$ is the one and only one common fixed point of H,I,C,P,Q and R.

6.3.2 Corollary: Suppose (X, d) is a complete bicomplex valued metric space and that H,C,P,Q,R are the self mappings on X satisfy (i) $H(X) \supseteq QR(X)$ and $H(X) \supseteq CP(X)$ (ii) $d(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1 d(H\varpi, H\vartheta) + \tau_2 d(H\varpi, CP\varpi) + \tau_3 d(H\vartheta, QR\vartheta) + \tau_4 d(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in X$, where τ_1,τ_2,τ_3 and τ_4 be non negative real numbers such that $\tau_1+\tau_2+\tau_3+2\tau_4 < 1$. (iii) Suppose that (QR,H) and (CP,H) are weakly compatible and (iv) (Q,R), (Q,H) (R,H),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P and H have one and only one common fixed point in X.

Proof: This result can be proved easily by substituting I = H in the above theorem.

6.4 Common coupled fixed point result in bicomplex valued metric space

Through this section, we prove a common coupled fixed point result for two self mappings in bicomplex valued metric space.

6.4.1 Theorem: Suppose (X, d) is a complete bicomplex valued metric space and h,k: X × X → X are two functions satisfy
$d(h(\varpi,\jmath),k(\rho,\sigma)) \preceq_{i_2} \tau_1 \frac{d(\varpi,\rho)+d(\jmath,\sigma)}{2} + \tau_2 \frac{d(\varpi,h(\varpi,\jmath))+d(\rho,\varpi)}{2} + \tau_3 \frac{d(\varpi,h(\varpi,\jmath))+d(\rho,k(\rho,\sigma))}{2}$
where $\varpi,\jmath,\rho,\sigma$ ∈X and τ_1,τ_2 and τ_3 are non negative integers such that $1 > \tau_1 + \tau_2 + \tau_3$. Then h and k contains one and only one common coupled fixed point in X×X.

Proof: Consider two arbitrary elements $\varpi_0,\jmath_0 \in X$. We define two sequences $\{\varpi_n\},\{\jmath_n\}$ such that $\varpi_{2l+1} = h(\varpi_{2l},\jmath_{2l}), \varpi_{2l+2} = k(\varpi_{2l+1},\jmath_{2l+1})$,
$\jmath_{2l+1} = h(\jmath_{2l},\varpi_{2l}), \jmath_{2l+2} = k(\jmath_{2l+1},\varpi_{2l+1})$, for l=0,1,2...

Now we consider,

$$\begin{aligned}
d(\varpi_{2l+1},\varpi_{2l+2}) &= d(h(\varpi_{2l},\jmath_{2l}),k(\varpi_{2l+1},\jmath_{2l+1})) \\
&\preceq_{i_2} \tau_1\frac{d(\varpi_{2l},\varpi_{2l+1})+d(\jmath_{2l},\jmath_{2l+1})}{2} + \tau_2\frac{d(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+d(\varpi_{2l+1},\varpi_{2l})}{2} \\
&+ \tau_3\frac{d(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+d(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2} \\
&= \tau_1\frac{d(\varpi_{2l},\varpi_{2l+1})+d(\jmath_{2l},\jmath_{2l+1})}{2} + \tau_2\frac{d(\varpi_{2l},\varpi_{2l+1})+d(\varpi_{2l+1},\varpi_{2l})}{2} \\
&+ \tau_3\frac{d(\varpi_{2l},\varpi_{2l+1})+d(\varpi_{2l+1},\varpi_{2l+2})}{2} \\
&= (\frac{\tau_1+2\tau_2+\tau_3}{2})d(\varpi_{2l},\varpi_{2l+1}) + (\frac{\tau_1}{2})d(\jmath_{2l},\jmath_{2l+1}) \\
&+ (\frac{\tau_3}{2})d(\varpi_{2l+1},\varpi_{2l+2})
\end{aligned}$$

i.e., $(2-\tau_3)d(\varpi_{2l+1}, \varpi_{2l+2}) \preceq_{i_2} (\tau_1+2\tau_2+\tau_3)d(\varpi_{2l}, \varpi_{2l+1}) + (\tau_1)\ d(\jmath_{2l}, \jmath_{2l+1})$ (6.4.1)

Again, we consider

$$\begin{aligned}
d(\jmath_{2l+1}, \jmath_{2l+2}) &= d(h(\jmath_{2l}, \varpi_{2l}), k(\jmath_{2l+1}, \varpi_{2l+1})) \\
&\preceq_{i_2} \tau_1 \frac{d(\jmath_{2l}^{'}, \jmath_{2l+1}) + d(\varpi_{2l}, \varpi_{2l+1})}{2} + \tau_2 \frac{d(\jmath_{2l}, h(\jmath_{2l}, \varpi_{2l})) + d(\jmath_{2l+1}, \jmath_{2l})}{2} \\
&+ \tau_3 \frac{d(\jmath_{2l}, h(\jmath_{2l}, \varpi_{2l}) + d(\jmath_{2l+1}, k(\jmath_{2l+1}, \varpi_{2l+1}))}{2} \\
&= \tau_1 \frac{d(\jmath_{2l}, \jmath_{2l+1}) + d(\varpi_{2l}, \varpi_{2l+1})}{2} + \tau_2 \frac{d(\jmath_{2l}, \jmath_{2l+1}) + d(\jmath_{2l+1}, \jmath_{2l})}{2} \\
&+ \tau_3 \frac{d(\jmath_{2l}, \jmath_{2l+1}) + d(\jmath_{2l+1}, \jmath_{2l+2})}{2} \\
&= (\frac{\tau_1 + 2\tau_2 + \tau_3}{2})d(\jmath_{2l}, \jmath_{2l+1}) + (\frac{\tau_1}{2})d(\varpi_{2l}, \varpi_{2l+1}) + (\frac{\tau_3}{2})d(\jmath_{2l+1}, \jmath_{2l+2})
\end{aligned}$$

i.e.,

$(2-\tau_3)d(\jmath_{2l+1}, \jmath_{2l+2}) \preceq_{i_2} (\tau_1+2\tau_2+\tau_3)d(\jmath_{2l}, \jmath_{2l+1}) + (\tau_1)\ d(\varpi_{2l}, \varpi_{2l+1})$ (6.4.2)

By adding the equations (6.4.1) and (6.4.2) we get

$d(\varpi_{2l+1}, \varpi_{2l+2}) + d(\jmath_{2l+1}, \jmath_{2l+2}) \preceq_{i_2} \eta[d(\varpi_{2l}, \varpi_{2l+1}) + d(\jmath_{2l}, \jmath_{2l+1})]$

where $\eta = \frac{2\tau_1+2\tau_2+\tau_3}{2-\tau_3}$ and $0 \leq \eta < 1$, since $1 > \tau_1 + \tau_2 + \tau_3$.

Similarly, we can easily show that

$d(\varpi_{2l+2}, \varpi_{2l+3}) + d(\jmath_{2l+2}, \jmath_{2l+3}) \preceq_{i_2} \eta[d(\varpi_{2l+1}, \varpi_{2l+2}) + d(\jmath_{2l+1}, \jmath_{2l+2})]$

Then, for any $l \in$ N, we get

$$\begin{aligned}
d(\varpi_{l+2}, \varpi_{l+1}) + d(\jmath_{l+2}, \jmath_{l+1}) &\preceq_{i_2} \eta[d(\varpi_{l+1}, \varpi_l) + d(\jmath_{l+1}, \jmath_l)] \\
&\preceq_{i_2} \eta^2[d(\varpi_l, \varpi_{l-1}) + d(\jmath_l, \jmath_{l-1})] \\
&\ldots\ldots\ldots \\
&\preceq_{i_2} \eta^{l+1}[d(\varpi_1, \varpi_0) + d(\jmath_1, \jmath_0)]
\end{aligned}$$

Now, we consider $m, l \in$ N and m $> l$, we get

$$\begin{aligned}
d(\varpi_m, \varpi_l) + d(\jmath_m, \jmath_l) &\preceq_{i_2} [d(\varpi_l, \varpi_{l+1}) + d(\jmath_l, \jmath_{l+2})] + [d(\varpi_{l+1}, \varpi_m) + d(\jmath_{l+1}, \jmath_m)] \\
&\preceq_{i_2} [d(\varpi_l, \varpi_{l+1}) + d(\jmath_l, \jmath_{l+1})] + [d(\varpi_{l+1}, \varpi_{l+2}) \\
&+ d(\jmath_{l+1}, \jmath_{l+2})] + \ldots\ldots + [d(\varpi_{m-1}, \varpi_m) + d(\jmath_{m-1}, \jmath_m)] \\
&\preceq_{i_2} [\eta^l + \eta^{l+1} + \eta^{l+2} + \ldots. + \eta^{m-1}][d(\varpi_1, \varpi_0) + d(\jmath_1, \jmath_0)] \\
&\preceq_{i_2} (\frac{\eta^l}{1-\eta})[d(\varpi_1, \varpi_0) + d(\jmath_1, \jmath_0)]
\end{aligned}$$

Since $0 \leq \eta < 1$, Then $d(\varpi_m, \varpi_l) \to 0$ & $d(\jmath_m, \jmath_l) \to 0$, as $l, m \to \infty$.

Hence $\{\varpi_n\}$ and $\{\jmath_n\}$ are two Cauchy sequences in X and so there exist $\varpi, \jmath \in$ X such that $(\varpi_n) \to \varpi$ and $(\jmath_n) \to \jmath$ as n$\to \infty$.

Now we consider

$$\begin{aligned} d(h(\varpi,\jmath),\varpi) &\preceq_{i_2} d(h(\varpi,\jmath),\varpi_{2l+2}) + d(\varpi_{2l+2},\varpi) \\ &= d(h(\varpi,\jmath),k(\varpi_{2l+1},s_{2l+1}) + d(\varpi_{2l+2},\varpi) \\ &\preceq_{i_2} \tau_1 \frac{d(\varpi,\varpi_{2l+1}) + d(\jmath,\jmath_{2l+1})}{2} + \tau_2 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi_{2l+1},\varpi)}{2} \\ &+ \tau_3 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2} + d(\varpi_{2l+2},\varpi) \\ &= \tau_1 \frac{d(\varpi,\varpi_{2l+1}) + d(\jmath,\jmath_{2l+1})}{2} + \tau_2 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi_{2l+1},\varpi)}{2} \\ &+ \tau_3 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi_{2l+1},\varpi_{2l+2})}{2} + d(\varpi_{2l+2},\varpi) \end{aligned}$$

Letting the limit as l $\to \infty$, then we get

$$\|d(h(\varpi,\jmath),\varpi)\| \leq (\frac{\tau_2+\tau_3}{2})\|d(h(\varpi,\jmath),\varpi)\|$$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|d(h(\varpi,\jmath),\varpi)\| = 0$.

Hence, $h(\varpi,\jmath) = \varpi$.

Similarly it can easily be shown that $h(\jmath,\varpi) = \jmath$.

Now we consider,

$$\begin{aligned} d(\varpi,k(\varpi,\jmath)) &= d(h(\varpi,\jmath),k(\varpi,\jmath)) \\ &\preceq_{i_2} \tau_1 \frac{d(\varpi,\varpi) + d(\jmath,\jmath)}{2} + \tau_2 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi,\varpi)}{2} \\ &+ \tau_3 \frac{d(\varpi,h(\varpi,\jmath)) + d(\varpi,k(\varpi,\jmath))}{2} \end{aligned}$$

i.e., $d(\varpi,k(\varpi,\jmath)) \preceq_{i_2} \frac{\tau_3}{2}\ d(\varpi,k(\varpi,\jmath))$

i.e., $\|d(\varpi,k(\varpi,\jmath))\| \leq \frac{\tau_3}{2}\ \|d(\varpi,k(\varpi,\jmath))\|$

i.e., $(1-\frac{\tau_3}{2})\ \|d(\varpi,k(\varpi,\jmath))\| \leq 0$, since $1 > \tau_1+\tau_2+\tau_3$.

Therefore, we get $\|d(\varpi,k(\varpi,\jmath))\| = 0$. Hence $k(\varpi,\jmath) = \varpi$.

Similarly, we can easily show that $k(\jmath,\varpi) = \varpi$.

Thus, $(\varpi,\jmath)$ is a common coupled fixed point of h and k.

Now we prove $(\varpi,\jmath)$ is unique.

Let (ℓ,υ) is any other common coupled fixed point of h and k.

Then $h(\ell,\upsilon) = k(\ell,\upsilon) = \ell$ and $h(\upsilon,\ell) = k(\upsilon,\ell) = \upsilon$.

Now we consider,

$$
\begin{aligned}
d(\varpi, \ell) &= d(h(\varpi, \jmath), k(\ell, \upsilon)) \\
&\preceq_{i_2} \tau_1 \frac{d(\varpi, \ell) + d(\jmath, \upsilon)}{2} + \tau_2 \frac{d(\varpi, h(\varpi, \jmath)) + d(\ell, \varpi)}{2} \\
&+ \tau_3 \frac{d(\varpi, h(\varpi, \jmath)) + d(\ell, k(\ell, \upsilon))}{2} \\
&= \tau_1 \frac{d(\varpi, \ell) + d(\jmath, \upsilon)}{2} + \tau_2 \frac{d(\varpi, \varpi) + d(\ell, \varpi)}{2} + \tau_3 \frac{d(\varpi, \varpi) + d(\ell, \ell)}{2} \\
&= (\frac{\tau_1 + \tau_2}{2}) d(\varpi, \ell) + \frac{\tau_1}{2} d(\jmath, \upsilon) \quad (6.4.3)
\end{aligned}
$$

Similarly, we can show that

$$
d(\jmath, \upsilon) \preceq_{i_2} (\frac{\tau_1 + \tau_2}{2}) d(\jmath, \upsilon) + \frac{\tau_1}{2} d(\varpi, \ell) \qquad (6.4.4)
$$

By adding the equations (6.4.3) and (6.4.4), we get

$d(\varpi, \ell) + d(\jmath, \upsilon) \preceq_{i_2} (\frac{2\tau_1 + \tau_2}{2})[d(\varpi, \ell) + d(\jmath, \upsilon)]$

i.e., $(1 - \frac{2\tau_1 + \tau_2}{2})[d(\varpi, \ell) + d(\jmath, \upsilon)] \preceq_{i_2} 0$.

Since $1 > \tau_1 + \tau_2 + \tau_3$,

we get $\|d(\varpi, \ell) + d(\jmath, \upsilon)\| \leq 0$.

So $d(\varpi, \ell) + d(\jmath, \upsilon) = 0$.

Hence, $\varpi = \ell$ and $\jmath = \upsilon$. i.e., $(\varpi, \jmath) = (\ell, \upsilon)$.

Hence $(\varpi, \jmath)$ is one and only one common coupled fixed point of h and k.

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List of Publications

1. Duduka Venkatesh, V.Naga Raju, "Some Fixed Point Results in Bicomplex Valued Metric Spaces," **Mathematics and Statistics(Scopus)**, Vol.10, No.6, pp. 1334-1339, 2022. DOI: 10.13189/ms.2022.100620.

2. Duduka Venkatesh, V.Naga Raju, "SOME FIXED POINT OUTCOMES IN Sb-METRIC SPACES USING (ψ, ϕ) - GENERALIZED WEAKLY CONTRACTIVE MAPS IN Sb-METRIC SPACES", **Global Journal Of Pure and Applied Mathematics**, ISSN 0973-1768 Volume 18, Number 2 (2022), pp. 753-770.

3. Duduka Venkatesh, V.Naga Raju, "Some fixed point results using (ψ, ϕ)- generalized almost weakly contractive maps in S-metric spaces", **Ratio Mathematica(UGC Care list)**, Online published, Vol.47(2023), Doi: 10.23755/rm.v4li0.855.

4. Duduka Venkatesh, V.Naga Raju, "SOME FIXED POINT RESULTS USING SIMULATION FUNCTION", **International Journal Of Analysis and Applications(SCOPUS and WOS)**, ISSN:2291-8639, Vol.21,No.8, 2023. Doi: 10.28924/2291-8639-21-2023-8.

5. Duduka Venkatesh, V.Naga Raju, "Some Fixed- Point Results in Sb- Metric Spaces," **Advances and Applications in Mathematical Sciences(UGC Care list)**, ISSN: 0974-6803, Vol. 22, Issue 4, pp.883-899, 2023.

Ratio Mathematica Volume 47, 2023

Some fixed point results using (ψ, ϕ)-generalized almost weakly contractive maps in S-metric spaces

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V.Naga Raju †

Abstract

Fixed point theorems have been proved for various contractive conditions by several authors in the existing literature. In this article, we define an (ψ, ϕ)- generalized almost weakly contractive map in S-metric spaces and prove an existence and uniqueness of fixed point of such maps. And also we deduce some existing results as special cases of our result. Moreover, we give an example in support of the results.

Keywords: Fixed point; generalized almost weakly contractive map; S-metric space;
2020 AMS subject classifications: 47H10,54H25 [1]

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[1] Received on September 7, 2022. Accepted on March 1, 2021. Published on June 30, 2023. doi: 10.23755/rm.v4li0.855. ISSN: 1592-7415. eISSN: 2282-8214.

Int. J. Anal. Appl. (2023), 21:8

International Journal of Analysis and Applications

Some Invariant Point Results Using Simulation Function

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Abstract. Through this article, we establish an invariant point theorem by defining generalized Z_s-contractions in relation to the simulation function in S-metric space. In this article, we generalized the results of Nihal Tas, Nihal Yilmaz Ozgur and N.Mlaiki. In addition to that, we bestow an example which supports our results.

1. Introduction

Fixed point is also known as an invariant point. Banach principle of contraction [2] on metric space plays very important role in the field of invariant point theory and non linear analysis. In 1922, Stefan Banach initiated the concept of contraction and established well known Banach contraction theorem. In the year 2006, B Sims and Mustafa [9], established theory on G-metric spaces, that is an extension of metric spaces and established some properties. Later, A.Aliouche, S.Sedghi and N.Shobe [13] initiated S-metric spaces, it is a generalization of G-metric spaces in the year 2012. In 2014, S.Radojevic, N.V.Dung and N.T.Hieu [4] proved by examples that S-metric space is not a generalization of G-metric space and vice versa. Invariant points of various contractive maps on S-metric spaces were studied in [[1], [3], [6]- [8], [11]]. In 2015, F.Khajasteh, Satish Shukla and S.Radenovic [5] introduced simulation function and the concept of Z-contration in relation to simulation function and proved an invariant point theorem which generalizes the Banach Contraction principle. Very recently, Murat Olgun, O.Bicer and T.Alyildiz [10] defined generalized Z-contraction in relation to the simulation function and proved an invariant point theorem.

Received: Dec. 3, 2022.

2020 *Mathematics Subject Classification.* 54H25, 47H09, 47H10.

Key words and phrases. Simulation function; Z-contraction; Fixed point; S-metric space.

https://doi.org/10.28924/2291-8639-21-2023-8

ISSN: 2291-8639

Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 18, Number 2 (2022), pp. 753-770

http://www.ripublication.com/gjpam.htm

Some Fixed Point Outcomes in S_b-Metric Spaces using (ϕ, ψ)-Generalized Weakly Contractive Maps in S_b-Metric Spaces

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Abstract

In this result, we define (ψ, ϕ) -generalized weakly contraction map in S_b-metric space. In the year 2017, B.K.Leta and G.V.R.Babu[3] defined (α, ψ, ϕ)-generalized weakly contractive maps in S-metric spaces and established the existence and uniqueness of fixed point theorem for such maps. By the motivation of B.K.Leta and G.V.R.Babu[3] results in S-metric spaces, we introduced the (ψ, ϕ) - generalized weakly contractive map in S_b-metric spaces and prove a existence and uniqueness of fixed point theorem. We also give an example to support of our result.

Keywords: Fixed point, S-metric space, S_b-metric space, (ψ, ϕ)- generalized weakly contracition map.
2010 MSC: 47H10, 54E50

1. INTRODUCTION

During 1922, Stefan Banach conceived the concept of contraction and established well known Banach contraction theorem. Banach Principle of contraction[9] on metric spaces is the paramount importance cause in the field of fixed points and non linear analysis. Literature's are brought out new outcomes that are related to prove the generalization of metric space and to acquire a refinement about the contractive

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Advances and Applications in Mathematical Sciences
Volume 22, Issue 4, February 2023, Pages 883-899

SOME FIXED-POINT RESULTS IN S_b-METRIC SPACES

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Abstract

In this paper, we establish some fixed point and common fixed-point theorems in S_b-metric spaces using implicit relation. The results presented in this paper extend and generalize several results from the existing literature.

1. Introduction

In 1906, Maurice Fréchet [4] introduced the concept of metric spaces. Later, in the year 1922, Stefan Banach [2] proved a very famous theorem called "Banach Fixed Point Theorem". In 2006, Z. Mustafa and B. Sims [5] introduced G-metric spaces. In 2012, Sedghi, Shobe and Aliouche [11] introduced S-metric spaces and they claimed that S-metric spaces are the generalization of G-metric spaces. But, later Dung, Hieu and Radojevic [3] have given examples that S-metric spaces are not the generalization of G-metric spaces or vice versa. Therefore, the collection of G-metric spaces and S-metric spaces are different. In 1989, I. A. Bakhtin [1] introduced b-metric spaces as a generalization of metric spaces. In 2016, N. Souayah, N. Mlaiki [12] introduced S_b-metric spaces as the generalizations of b-metric spaces and S-metric spaces. But, very recently Tas and Ozur [6] studied some relations between S_b-metric spaces and some other metric spaces. S. Sedghi and N. V. Dung [9] introduced an implicit relation to investigate some fixed-

2020 Mathematics Subject Classification: 47H10, 54H25.

Keywords: Fixed point, Common fixed point, implicit relation, S_b-metric space.

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Received September 15, 2022; Revised December 18, 2022; Accepted December 19, 2022

Mathematics and Statistics 10(6): 1334-1339, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100620

Some Fixed Point Results in Bicomplex Valued Metric Spaces

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Received August 10, 2022; Revised October 12, 2022; Accepted October 25, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Duduka Venkatesh, V.Naga Raju, "Some Fixed Point Results in Bicomplex Valued Metric Spaces," Mathematics and Statistics, Vol.10, No.6, pp. 1334-1339, 2022. DOI: 10.13189/ms.2022.100620*

***(b)**: Duduka Venkatesh, V.Naga Raju (2022). Some Fixed Point Results in Bicomplex Valued Metric Spaces. Mathematics and Statistics, 10(6), 1334-1339. DOI: 10.13189/ms.2022.100620*

Abstract Fixed points are also called as invariant points. Invariant point theorems are very essential tools in solving problems arising in different branches of mathematical analysis. In the present paper, we establish three unique common invariant point theorems using two self-mappings, four self-mappings and six self-mappings in the bicomplex valued metric space. In the first theorem, we generate a common invariant point theorem for four self-mappings by using weaker conditions such as weakly compatible, generalized contraction and (CLR_{AB}) property. Then, in the second theorem, we generate a common invariant point theorem for six self-mappings by using inclusion relation, generalized contraction, weakly compatible and commuting maps. Further, in the third theorem, we generate a common coupled invariant point for two self mappings using different contractions in the bicomplex valued metric space. The above results are the extention and generalization of the results of [11] in the Bicomplex metric space. Moreover, we provide an example which supports the results.

Keywords Bicomplex Valued Metric Space, Common Fixed Point, Coupled Fixed Point, CLR Property, Weakly Compatible Mappings

---

1 Introduction

The concepts of bicomplex numbers and tricomplex numbers were introduced in the year 1892 by Segre[1]. Complex valued metric spaces are introduced by Azam et al.[2], in the year 2011 and some results were studied for such spaces. Very recently, the bicomplex valued metric space was introduced by Cho et al.[5] and some fixed point results were obtained. In the year 2019, Jebril, Datta, Sarkar and Biswas [6] derived some fixed point outcomes using rational contractions in bicomplex valued metric space.

Imdad et al.[8] introduced a new notion, called CLR-property for self maps in 2012. Afterwards, by using it several mathematicians obtained some fixed point results ([3],[4],[9] and [10]). The main purpose of this work is to prove some invariant point outcomes using various contractions for four self mappings, six self mappings and coupled invariant point theorems using weakly compatibility, CLR_{AB} property and commuting maps in bicomplex valued metric spaces.

2 Preliminaries

We denote C_0 = R(Real numbers), C_1 = C(Complex numbers) and C_2 = Set of all bicomplex numbers.
Let $\varpi, \vartheta \in C_1$, then we define a partial order $\preceq$ on C_1 as:
$\varpi \preceq \vartheta \iff Re(\varpi) \leq Re(\vartheta)$ and $Im(\varpi) \leq Im(\vartheta)$.
Also $\varpi \prec \vartheta$ if $Re(\varpi) < Re(\vartheta)$ and $Im(\varpi) < Im(\vartheta)$.
Segre[1] defined the bicomplex number as:

$$\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2,$$

where $b_1, b_2, b_3, b_4 \in C_0$, and i_1, i_2 are the independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$,
we defined C_2 as:

$$C_2 = \{\zeta : \zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2, b_1, b_2, b_3, b_4 \in C_0\},$$

i.e.,

$$C_2 = \{\zeta : \zeta = \varpi + i_2 \vartheta, \varpi, \vartheta \in C_1\}$$

Existence and Uniqueness of Fixed and Common Fixed Points for Different Contractions in Various Spaces

by Venkatesh D.

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